TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 222, 1976

SINGULARITIES IN THE NILPOTENT SCHEME OF A CLASSICAL GROUP

BY

WIM HESSELINK

ABSTRACT. If (X, x) is a pointed scheme over a ring k, we introduce a (generalized) partition $\mathrm{ord}(x, X/k)$. If G is a reductive group scheme over k, the existence of a nilpotent subscheme $N(G)$ of $\mathrm{Lie}(G)$ is discussed. We prove that $\mathrm{ord}(x, N(G)/k)$ characterizes the orbits in $N(G)$, their codimension and their adjacency structure, provided that G is Gl_n, or Sp_n and $1/2 \in k$. For SO_n only partial results are obtained. We give presentations of some singularities of $N(G)$. Tables for its orbit structure are added.

Introduction. Let G be a reductive algebraic group over a field of characteristic p. Let $\mathfrak{g}$ be its Lie-algebra and $N(G)$ the closed subset of the nilpotent elements of $\mathfrak{g}$, cf. [19]. The G-orbits in $N(G)$ are characterized by weighted Dynkin diagrams, cf. [20, III]. Consider the following question. *Is it possible to classify the orbits in $N(G)$ using only the local structure of the variety $N(G)$*? We prove in (4.3) that the answer is positive if G is Gl_n or if G is Sp_n and $p \neq 2$.

To this end we introduce a local invariant "ord" for any pointed scheme in §1. We develop the theory of $N(G)$ over an arbitrary ground ring k in §2. In §3 we restrict our attention to the classical group schemes. Using a cross section we obtain information about the orbit structure of $N(G)$. Our main theorem (4.2) relates $\mathrm{ord}(x, N(G)/k)$ to the Jordan normal form of the nilpotent endomorphism induced by x in the classical representation.

This paper is a condensed version of [13]. The author wishes to express his gratitude to his thesis adviser, Professor T. A. Springer.

Conventions and notations. The cardinality of a set V is denoted by $\# V$ Any infinite cardinal is represented by ∞. If x is a real number then $[x]$ is the greatest integer in x. All rings are commutative with 1. Let M be a module over a ring A. If M is free the rank of M is denoted by $\mathrm{rg}_A M$. An element $a \in A$ is called M-regular if $a: M \longrightarrow M$ is injective. Let $\mathrm{a} = (a_1, \ldots, a_r)$ be a

Received by the editors March 18, 1975.

AMS (MOS) subject classifications (1970). Primary 14B05, 14L15; Secondary 05A17, 10C30, 13H15, 20G35.

Key words and phrases. Partitions, linear extension of a local ring, regular sequence, classical group scheme, orbit, cross section, nilpotent variety, Kleinian singularity

sequence in A. The ideal generated by a is denoted by $\langle a \rangle$. The sequence is called M-regular if a_i is $(M/\langle (a_j)_{j<i} \rangle M)$-regular for all i, cf. [12, 0_{IV} 15.1].

Unless stated otherwise k is an arbitrary ground ring. General references for schemes and group schemes are [11], [12] and [8]. If we consider a k-scheme as a functor from k-algebras to sets, cf. [11, p. 17], then the letter R is used to denote an arbitrary k-algebra. If X is a k-scheme and R is a k-algebra then $X_{(R)}$ is the R-scheme $X \otimes_k R$. If X is an affine scheme then its coordinate ring is denoted by $A(X)$. If A is a local ring its maximal ideal is denoted by $\mathfrak{m}_A$ and its residue field by $\mathrm{k}(A)$. If X is a scheme and $x \in X$ then we write $\mathfrak{m}_x := \mathfrak{m}_A$ and $\mathrm{k}(x) := \mathrm{k}(A)$ where $A := \mathcal{O}_{X,x}$.

1. **A near-partition for a local k-algebra.**

(1.1) A *near-partition* λ is a subset of $\mathbf{N}^2$ such that if $(m, n) \in \lambda$ and $i \leqslant m$ and $j \leqslant n$ then $(i, j) \in \lambda$. The set of near-partitions is denoted by NP. The duality mapping D: $NP \longrightarrow NP$ is induced by $(i, j) \longmapsto (j, i)$. The set NP is ordered by $\lambda \leqslant \mu$ if and only if $\lambda \subset \mu$. We write $|\lambda| := \# \lambda$. A near-partition λ is called a *partition* if $|\lambda| < \infty$. The set of partitions is denoted by P.

If $\lambda \in NP$, the nonincreasing sequences λ^* and λ_* in $\{0\} \cup \mathbf{N} \cup \{\infty\}$ are defined by

$$\lambda^n \geqslant i \Leftrightarrow (n, i) \in \lambda \Leftrightarrow \lambda_i \geqslant n.$$

Clearly $\lambda_i = (D\lambda)^i = \sup \{n \in \mathbf{N} | \lambda^n \geqslant i\}$, and dually. A near-partition λ is completely determined by its sequence λ^* (or λ_*). We write $\lambda_* = (\lambda_1, \ldots, \lambda_r)$ if $\lambda_i = 0$ for $i > r$. If $\lambda, \mu \in NP$, we define $\lambda + \mu \in NP$ by $(\lambda + \mu)^n := \lambda^n + \mu^n$, where $x + \infty := \infty + x := \infty$ for all x. If $\lambda_* = (\lambda_1, \ldots, \lambda_r)$ and $\mu_* = (\mu_1, \ldots, \mu_s)$ then $(\lambda + \mu)_*$ is the sequence obtained by ordering $(\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_s)$, see [9, Proposition 6].

(1.2) Definition. A *linear extension* over a ring k is a surjective morphism ϵ: $E \longrightarrow A$ of local k-algebras such that $\mathfrak{m}_E \ker(\epsilon) = 0$. Its near-partition $\mathrm{ord}(\epsilon)$ is defined by

$$\mathrm{ord}^n(\epsilon) := \mathrm{rg}_{\mathrm{k}(E)}(\ker(\epsilon) \cap \mathfrak{m}_E^{n+1}).$$

A linear extension ϵ: $E \longrightarrow A$ is called *versal* over k if for any linear extension ζ: $F \longrightarrow B$ over k and any local k-morphism ϕ: $A \longrightarrow B$ there exists a (clearly local) k-morphism γ: $E \longrightarrow F$ with $\zeta \circ \gamma = \phi \circ \epsilon$, see diagram (i).

(1.3) Proposition. *Let diagram* (i) *be a commutative diagram of k-algebras such that ϵ and ζ are linear extensions, that ϕ is a flat local morphism and that* $\mathfrak{m}_A B = \mathfrak{m}_B$. *Then we have* $\mathrm{ord}(\epsilon) \geqslant \mathrm{ord}(\zeta)$.

(i)
$$\begin{array}{ccc} F & \xrightarrow{\zeta} & B \\ {\scriptstyle\gamma}\uparrow & & \uparrow{\scriptstyle\phi} \\ E & \xrightarrow{\epsilon} & A \end{array}$$

PROOF. Let $n \in \mathbf{N}$. We prove that $\mathrm{ord}^n(\epsilon) \geqslant \mathrm{ord}^n(\zeta)$. It suffices to prove that the ideal $\ker(\zeta) \cap \mathfrak{m}_F^{n+1}$ is generated by the image of $\ker(\epsilon) \cap \mathfrak{m}_E^{n+1}$. We may assume that $\ker(\epsilon) \cap \mathfrak{m}_E^{n+1} = 0$. Now the mapping $\mathfrak{m}_E^{n+1} \longrightarrow A$ induced by ϵ is an injection of A-modules. Since B is flat over A, it follows that $\mathfrak{m}_E^{n+1} \otimes_E B \longrightarrow B$ is injective and hence that $\mathrm{Tor}^E(E/\mathfrak{m}_E^{n+1}, B) = 0$. This implies injectivity of

$$\ker(\zeta) \otimes_E (E/\mathfrak{m}_E^{n+1}) \longrightarrow F \otimes_E (E/\mathfrak{m}_E^{n+1})$$

so that $\ker(\zeta) \cap \mathfrak{m}_E^{n+1} F = \mathfrak{m}_E^{n+1} \ker(\zeta) = 0$. On the other hand $\mathfrak{m}_A B = \mathfrak{m}_B$ implies that $\mathfrak{m}_E F + \ker(\zeta) = \mathfrak{m}_F$, so that $\mathfrak{m}_E^{n+1} F = \mathfrak{m}_F^{n+1}$. This proves $\ker(\zeta) \cap \mathfrak{m}_F^{n+1} = 0$.

(1.4) Let A be a local k-algebra. If ϵ: $E \longrightarrow A$ is a versal linear extension over k then (1.3) implies that $\mathrm{ord}(\epsilon) \geqslant \mathrm{ord}(\zeta)$ for any linear extension $\zeta: F \longrightarrow A$ over k. On the other hand there exists a versal linear extension ϵ: $E \longrightarrow A$ over k. In fact, write $A = R/A$ where R is some polynomial k-algebra. Let M be the ideal in R such that $\mathfrak{m}_A = M/J$. Then $R/MJ \longrightarrow A$ is a versal linear extension over k, compare [15, p. 37]. Now we can give the following:

DEFINITION. $\mathrm{ord}(A/k) := \mathrm{ord}(\epsilon)$ where ϵ: $E \longrightarrow A$ is some (or any) versal linear extension over k.

EXAMPLE. Let k be a field. Put $H := k[T_1, \ldots, T_m]$. Let $\mathrm{a} = (a_1, \ldots, a_r)$ be a sequence in H. Let a_i be homogeneous of degree $1 + \lambda_i$ where λ is a partition with $\lambda_{r+1} = 0$. Assume that the ideal $\langle \mathrm{a} \rangle$ is not generated by a strict subsequence of a. Consider the local ring $A := (H/\langle \mathrm{a} \rangle)_{\mathfrak{p}}$ where $\mathfrak{p} = \langle T_1, \ldots, T_m \rangle$. Then $\mathrm{ord}(A/k) = \lambda$.

In fact ϵ: $(H/\mathfrak{p}\langle a \rangle)_{\mathfrak{p}} \longrightarrow A$ is a versal linear extension over k and $\mathrm{ord}^n(\epsilon) = \#\{i \mid \lambda_i \geqslant n\} = \lambda^n$.

(1.5) PROPOSITION. *Let A be a local k-algebra and R a k-algebra. Assume that A or R is flat over k. Let* $\mathfrak{p} \in \mathrm{Spec}(A \otimes_k R)$ *contract to* $\mathfrak{m}_A$. *Then* $\mathrm{ord}(A/k) \leqslant \mathrm{ord}((A \otimes_k R)_{\mathfrak{p}}/R)$.

PROOF. Let ϵ: $E \longrightarrow A$ be a versal linear extension over k. Put $I := \ker(\epsilon)$. Let $\mathfrak{q} \in \mathrm{Spec}(E \otimes R)$ be the inverse image of $\mathfrak{p}$. Since A or R is flat over k, $(I \otimes R)_{\mathfrak{q}}$ is an ideal in $(E \otimes R)_{\mathfrak{q}}$. Put $F := (E \otimes R)_{\mathfrak{q}}/\mathfrak{q}(I \otimes R)_{\mathfrak{q}}$, so that ζ: $F \longrightarrow (A \otimes R)_{\mathfrak{p}}$ is a linear extension over R. One verifies that $I \otimes_{\mathrm{k}(E)} \mathrm{k}(F) \longrightarrow \ker(\zeta)$ is injective and hence that $\mathrm{ord}(\epsilon) \leqslant \mathrm{ord}(\zeta)$. This suffices.

(1.6) PROPOSITION. *Let A be a local k-algebra, $\mathrm{x} = (x_1, \ldots, x_m)$ an A-regular sequence in $\mathfrak{m}_A$ and f a nonzero element of $\langle \mathrm{x} \rangle$. Put $B = A/\langle f \rangle$ and $C = A/\langle \mathrm{x} \rangle$. Let $r \in \mathrm{N}$ and let ρ be a partition with $\rho_* = (r-1)$.*

(a) *If $f \in \mathfrak{m}_A^r$ then* $\rho + \mathrm{ord}(A/k) \leqslant \mathrm{ord}(B/k)$.

(b) *If $f \notin \mathfrak{m}_A^{r+1}$ then* $\mathrm{ord}(B/k) \leqslant \rho + \mathrm{ord}(C/k)$.

PROOF. Let ϵ: $E \longrightarrow A$ be a versal linear extension over k. Put $I = \ker(\epsilon)$. Choose $y_i \in E$ with $\epsilon(y_i) = x_i$ and $g \in E$ with $\epsilon(g) = f$. Put $F := \dot{E}/g\mathfrak{m}_E$ and $G := E/\mathfrak{m}_E\langle \mathrm{y} \rangle$. The linear extensions ζ: $F \longrightarrow B$ and η: $G \longrightarrow C$ are versal over k. Since x is a regular sequence, we have $I \cap \langle \mathrm{y} \rangle = 0$. So the induced mappings $I \longrightarrow \ker(\zeta)$ and $I \longrightarrow \ker(\eta)$ are injective. This implies that $\mathrm{ord}(\epsilon) \leqslant \mathrm{ord}(\zeta)$ and $\mathrm{ord}(\epsilon) \leqslant \mathrm{ord}(\eta)$.

(a) Now it suffices to prove:

(∗) If $n < r$ then $1 + \mathrm{ord}^n(\epsilon) = \mathrm{ord}^n(\zeta)$.

We may assume that $g \in \mathfrak{m}_E^{n+1}$. The cokernel of the injection $I \cap \mathfrak{m}_E^{n+1} \longrightarrow \ker(\zeta) \cap \mathfrak{m}_F^{n+1}$ is isomorphic to $\langle g \rangle / g\mathfrak{m}_E$; this proves (∗).

(b) By (∗) it suffices to prove: If $f \notin \mathfrak{m}_A^{n+1}$ then $\mathrm{ord}^n(\zeta) \leqslant \mathrm{ord}^n(\eta)$. We may assume $g \in \langle \mathrm{y} \rangle$. Since $f \notin \mathfrak{m}_A^{n+1}$ we have $g \notin \mathfrak{m}_E^{n+1}$. Using that $I \cap \langle \mathrm{y} \rangle = 0$, one shows that the mapping $\ker(\zeta) \cap \mathfrak{m}_F^{n+1} \longrightarrow \ker(\eta) \cap \mathfrak{m}_G^{n+1}$ is injective.

REMARK. Usually (1.6) (a) is applied in the situation where f itself is A-regular, $m = 1$ and $x_1 = f$.

(1.7) If X is a k-scheme and $x \in X$ then (X, x) is called a *pointed k-scheme.* We define $\mathrm{ord}(x, X/k) := \mathrm{ord}(\mathcal{O}_{X,x}/k)$. Pointed k-schemes (X, x) and (Y, y) are called *smoothly equivalent* if there are smooth k-morphisms f: $Z \longrightarrow X$, g: $Z \longrightarrow Y$ and a point $z \in Z$ with $f(z) = x$, $g(z) = y$. This is an equivalence relation on the class of pointed k-schemes, to be denoted by $(X, x) \sim (Y, y)$. See [12, IV 17] for the definition and the basic properties of smooth morphisms.

THEOREM. *If $(X, x) \sim (Y, y)$ then* $\mathrm{ord}(x, X/k) = \mathrm{ord}(y, Y/k)$.

PROOF. We may assume that there is a smooth k-morphism f: $X \longrightarrow Y$ with $f(x) = y$.

Using the regularity of the noetherian local ring $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ and the arguments of the proof of [12, IV 19.2.9], we construct a subscheme Z of X containing x such that $\mathcal{O}_{Z,x} = \mathcal{O}_{X,x}/\langle \mathrm{a} \rangle$ where a is an $\mathcal{O}_{X,x}$-regular sequence, that $\mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{Z,x}$ is flat and that $\mathfrak{m}_y\mathcal{O}_{Z,x}$ is the maximal ideal of $\mathcal{O}_{Z,x}$. By (1.6) (a) we have $\mathrm{ord}(x, X/k) \leqslant \mathrm{ord}(x, Z/k)$. Using (1.3) one proves that $\mathrm{ord}(x, Z/k) \leqslant \mathrm{ord}(y, Y/k)$.

We may assume that $Y = \mathrm{Spec}\, A$ and $y = \mathfrak{m}_A$ where A is a local k-algebra. Choose a versal linear extension ϵ: $E \longrightarrow A$ over k. By [12, IV 18.1.1] there is a smooth E-algebra R such that $\mathrm{Spec}(A \otimes_E R)$ is isomorphic to an open neighbourhood of x in X. So $\mathcal{O}_{X,x} \cong (A \otimes_E R)_{\mathfrak{p}}$ for some $\mathfrak{p} \in \mathrm{Spec}(A \otimes_E R)$ contracting to $\mathfrak{m}_A$. By (1.5) we have $\mathrm{ord}(A/E) \leqslant \mathrm{ord}((A \otimes_E R)_{\mathfrak{p}}/R)$. It is easy to see that this implies $\mathrm{ord}(y, Y/k) \leqslant \mathrm{ord}(x, X/k)$.

(1.8) The following remark will not be used in the sequel. For proofs and details we refer to [13].

REMARK. Let A be a noetherian local k-algebra. Then $\mathrm{ord}(A/k)$ is a partition. It is equal to $\mathrm{ord}(\hat{A}/k)$ where $\hat{A}$ is the completion of A. If k is noetherian regular and A is of essentially finite type over k, then $\mathrm{ord}(A/k) = \mathrm{ord}(A/\mathbf{Z})$. A is regular if and only if $\mathrm{ord}(A/\mathbf{Z}) = 0$. If $A = R/J$ where J is an ideal in a noetherian regular local ring R, then $\mathrm{ord}(A/\mathbf{Z})$ is determined by the sequence $\nu_*(J)$,cf. [14, p. 209].

2. The nilpotent scheme.

(2.1) Consider an action h of an affine group scheme G on an affine scheme X over k. We have the morphisms h, pr_2: $G \times_k X \rightrightarrows X$. The *orbit* Gx of $x \in X$ is defined as the subset $h(\mathrm{pr}_2^{-1}(x))$ of X. Let V be a subscheme of X. Let U be the open set where the induced morphism h^V: $G \times_k V \longrightarrow X$ is smooth. V is called a *cross section at* x if $x \in V$ and $e_{(V)}(x) \in U$. Here $e_{(V)}$: $V \longrightarrow G \times_k V$ is induced by the unit $e \in G(k)$. The subscheme V is called a *global cross section* if $U \longrightarrow \mathrm{Spec}(k)$ is surjective. V is called an *invariant subscheme* if the morphism h^V factorizes over V.

Let $A(X)^G$ be the equalizer of the comorphisms $A(X) \rightrightarrows A(G) \otimes_k A(X)$. If Y is an affine k-scheme, a G-invariant k-morphism f: $X \longrightarrow Y$ corresponds to a comorphism $A(Y) \longrightarrow A(X)$ factorizing over $A(X)^G$. We define the *affine quotient* of the action by $[X/G] := \mathrm{Spec}(A(X)^G)$. It is called *universal* if the induced morphism $[X_{(R)}/G_{(R)}] \longrightarrow [X/G]_{(R)}$ is an isomorphism for any k-algebra R.

REMARKS. (a) Let G be smooth over k. Then pr_2 and h are smooth morphisms. If $x' \in Gx$ then $(X, x) \sim (X, x')$, cf. (1.7). If V is a cross section at x then $(X, x) \sim (V, x)$.

(b) The condition, that the affine quotient $[X/G]$ is universal, is a local condition on $\mathrm{Spec}(k)$ for the topology $(f\,p\,q\,c)$, cf. [8, IV], see [13, p. 38]. If k is a field any affine quotient is universal.

(c) Other types of quotients are discussed in [17, p. 3].

(2.2) PROPOSITION. *Assume in* (2.1) *that the morphism* $X \longrightarrow \mathrm{Spec}(k)$ *is smooth and irreducible cf.* [12, IV 4.5.5] *,and that V is affine and a global cross section.*

(a) *The morphism* $A(X)^G \longrightarrow A(V)$ *is injective.*

(b) *If* $A(X)^G \longrightarrow A(V)$ *is bijective then* $[X/G]$ *is universal.*

PROOF. (a) Consider a nonzero $f \in A(X)^G$. Assume that $f|V = 0$. There is a commutative diagram (i), so we have $f \circ h^V = 0$.

$$\text{(i)} \qquad \begin{array}{ccc} G \times_k V & \xrightarrow{\text{pr}_2} & V \\ {\scriptstyle h^V}\downarrow & & \downarrow{\scriptstyle f|V} \\ X & \xrightarrow{f} & \text{Spec}(k[T]) \end{array}$$

The morphism h^V is flat on U, so $h^V(U)$ is an open subset of X with $f|h^V(U) = 0$. Since $f \neq 0$, there is a generic point x of $\text{Supp}(f\mathcal{O}_X)$. Let $\mathfrak{p} \in \text{Spec}(k)$ be the image of x. Let ξ be the unique generic point of $X \otimes_k \text{k}(\mathfrak{p})$. As $h^V(U) \longrightarrow \text{Spec}(k)$ is surjective we have $\xi \in h^V(U)$ and hence $x \neq \xi$. Since $\mathcal{O}_{X,x} \otimes_k \text{k}(\mathfrak{p})$ is regular there is an $\mathcal{O}_{X,x} \otimes_k \text{k}(\mathfrak{p})$-regular element $t \in \mathfrak{m}_x$. By [12, IV 11.3.7], t is $\mathcal{O}_{X,x}$-regular. It is easy to see that this contradicts the choice of x. The argument used here was suggested by P. Deligne.

(b) Let R be a k-algebra. We have to prove that u: $A(X)^G \otimes R \longrightarrow A(X_{(R)})^{G(R)}$ is bijective. As the assumptions of (a) are stable under base-change, the morphism v: $A(X_{(R)})^{G(R)} \longrightarrow A(V) \otimes R$ is injective by (a). So it suffices to observe that $v \circ u$ is bijective.

(2.3) Let G be a smooth affine group scheme over k. Recall that the Lie-algebra $\text{Lie}(G)$ is defined as the group functor such that $\text{Lie}(G)(R)$ is the (additively written) kernel of the morphism $G(R[\delta]/\langle\delta^2\rangle) \longrightarrow G(R)$ induced by $\delta \mapsto 0$ where R is a k-algebra. $\text{Lie}(G)$ is a smooth affine group scheme, in fact a vector bundle. There is a canonical action of G on $\text{Lie}(G)$. If R is a k-algebra then $\text{Lie}(G)_{(R)} = \text{Lie}(G_{(R)})$. See [8, II 4]. Usually we write $\mathfrak{g} := \text{Lie}(G)$.

If K is a field over k, a section $x \in \mathfrak{g}(K) = \text{Lie}(G_{(K)})(K)$ is nilpotent if and only if its image is a nilpotent endomorphism of F for some (or any) immersion of $G_{(K)}$ in a K-group $Gl(F)$, cf. [2, p. 151]. A point $x \in \mathfrak{g}$ is called *nilpotent* if the corresponding section $x \in \mathfrak{g}(\text{k}(x))$ is nilpotent.

(2.4) DEFINITION. Let G be a reductive group scheme over k, cf. [8, XIX 2.7]. If the affine quotient $[\mathfrak{g}/G]$ is universal, cf. (2.1), then we define the *nilpotent scheme* $N(G) := p^{-1}p(0)$ where $0 \in \mathfrak{g}(k)$ is the zero section and p: $\mathfrak{g} \longrightarrow [\mathfrak{g}/G]$ is the quotient morphism.

PROPOSITION. *Let $N(G)$ be defined.*

(a) *$N(G)$ is a G-invariant closed subscheme of $\mathfrak{g}$.*

(b) *If R is a k-algebra then $N(G_{(R)}) = N(G)_{(R)}$.*

(c) *A point $x \in \mathfrak{g}$ is nilpotent if and only if $x \in N(G)$.*

PROOF. (a) is trivial. (b) is a consequence of the assumption that $[\mathfrak{g}/G]$ is universal. (c) By (b) we may assume that k is a field and that $x \in \mathfrak{g}(k)$. Now it is well known. The "if-part" follows from Cayley-Hamilton by an embedding of G in some $Gl(F)$. The "only-if-part" is a consequence of the following

LEMMA. *Let G be a reductive k-group over a field k. If $x \in \mathfrak{g}(k)$ has the additive Jordan decomposition $x = x_s + x_n$, then x_s is in the closure of the orbit Gx.*

PROOF. Adapt [22, (4.4)] or [21, p. 92].

(2.5) Let G be a reductive group scheme over k. By (2.1)(b) the existence of a nilpotent scheme is a local condition on Spec(k) for the topology $(f\,p\,q\,c)$. So we assume that G is of constant type (cf. [8, XXII 2.7]) with specified root system $\mathcal{R} = (M, R, \rho)$, i.e. a root system R in a given lattice M (cf. [7, p. 287]). Let t be the torsion index (cf. [7, p. 294]). Let f be the connection index (cf. [4, p. 167]). Consider the following conditions:

(i) $t^{-1} \in k$ and if $R \cap 2M \neq \varnothing$ then $1/2 \in k$, cf. [7, p. 296].

(ii) $t^{-1}f^{-1} \in k$.

(iii) If $\mathcal{R}$ has a component of type A_l then $(l+1)^{-1} \in k$, of type B_l, C_l, D_l, G_2 then $1/2 \in k$, of type E_6, E_7, F_4 then $1/6 \in k$, of type E_8 then $1/30 \in k$.

The conditions (ii) and (iii) are equivalent and imply (i).

(2.6) THEOREM. *Let G be as in* (2.5) *satisfying condition* (i).

(a) *The affine quotient $[\mathfrak{g}/G]$ is universal. The quotient morphisms $p\colon \mathfrak{g} \longrightarrow [\mathfrak{g}/G]$ is flat. $N(G)$ is defined and flat over k.*

(b) *Let T be a maximal torus of G with Weyl group W, cf.* [8, XXII 3]. *Put* $\mathfrak{t} := \mathrm{Lie}(T)$. *The affine quotient $[\mathfrak{t}/W]$ is universal. The canonical morphism $[\mathfrak{t}/W] \longrightarrow [\mathfrak{g}/G]$ is an isomorphism.*

(c) *Assume that* (2.5) (ii) *holds. Let $\pi\colon\ G \longrightarrow \mathrm{ad}(G)$ be the projection onto the adjoint group, cf.* [8, XXII 4.3]. *Then $N(\mathrm{ad}(G))$ is defined and equal to $N(G)$.*

PROOF. (1) We may assume that G is split with respect to a (resp. the) maximal torus T, cf. [8, XXII 2.3]. Now $T = D_S(M)$ and $A(\mathfrak{t}) = S(M) \otimes k$. The group scheme W is the constant group scheme associated to the abstract Weyl group of $\mathcal{R}$. By (2.5) (i) and [7, pp. 295, 296] the affine quotient $[\mathfrak{t}/W]$ is universal and the quotient morphism $\mathfrak{t} \longrightarrow [\mathfrak{t}/W]$ is flat.

(2) By [8, XIII 5.1] and [12, IV 17.8.3] the subscheme $\mathfrak{t}$ is a global cross section for the action of G on $\mathfrak{g}$, cf. (2.1). By (2.2) this implies that $A(\mathfrak{g})^G \longrightarrow A(\mathfrak{t})^W$ is injective.

(3) We may assume that $k = \mathbf{Z}[1/m]$, cf. [8, XXV 1]. It follows from [20, II 3.17′] and [22, p. 220] that $A(\mathfrak{g})^G \otimes_k \mathbf{Q} \longrightarrow A(\mathfrak{t})^W \otimes_k \mathbf{Q}$ is bijective. Consider $a \in A(\mathfrak{t})^W$. There is $a_1 \in A(\mathfrak{g})^G$ and a nonzero $s \in k$ with $a_1 | \mathfrak{t} = sa$. Put $R = k/\langle s\rangle$. Now $a_1 \otimes 1_R | \mathfrak{t}_{(R)} = 0$, so by (2) we have $a_1 \otimes 1_R = 0$ in

$A(\mathfrak{g}) \otimes R$. So there is $a_2 \in A(\mathfrak{g})$ with $a_1 = sa_2$. Since s is $A(G) \otimes A(\mathfrak{g})$-regular we have $a_2 \in A(\mathfrak{g})^G$. Since s is $A(\mathfrak{t})$-regular we have $a = a_2|\mathfrak{t}$. This proves that $A(\mathfrak{g})^G \longrightarrow A(\mathfrak{t})^W$ is bijective. So we have proved (b).

(4) With (b) and (2) one proves that $[\mathfrak{g}/G]$ is universal in the same way as in (2.2) (b). Let U be the open subset of $\mathfrak{g}$ where p is flat. Since $\mathfrak{t} \longrightarrow [\mathfrak{g}/G]$ is flat by (1) and (b), and $\mathfrak{t} \subset \mathfrak{g}$ is a regular immersion, we have $\mathfrak{t} \subset U$ by [12, 0_{IV} 15.1.16]. As U is G-invariant this implies $U = \mathfrak{g}$ by the lemma in (2.4). The other assertions of (a) follow immediately.

(5) In the notations of [8, XXII], condition (2.5) (ii) implies that the central isogenies $G \longrightarrow \mathrm{corad}(G) \otimes ss(G)$ and $ss(G) \longrightarrow \mathrm{ad}(G)$ are étale morphisms, by [8, VIII 2.1] and [8, XXI 6.5]. So we have an isomorphism

$$A(\mathrm{Lie}(\mathrm{ad}(G)))^{\mathrm{ad}(G)} \otimes A(\mathrm{Lie}(\mathrm{corad}(G))) \cong A(\mathfrak{g})^G.$$

With this isomorphism one proves (c).

REMARKS. (i) Assume that the order of the Weyl group is invertible in k. By [22, (6.9)] the morphism p is normal cf. [12, IV 6.8.1]. (ii) If $l \geqslant 2$ there is a semisimple group scheme G of type D_l over $\mathbf{Z}$ such that $[\mathfrak{g}/G]$ is not universal.

(2.7) COROLLARY. *Let G be as in* (2.6). *Let $d_1, \ldots, d_r$ be the degrees of* R. *Consider the partition λ defined by $\lambda_i := d_{r+1-i} - 1$ if $i \leqslant r$ and $\lambda_{r+1} := 0$. Let x be a point of the zero section of* $\mathfrak{g}$. *Then* $\mathrm{ord}(x, N(G)/k) = \lambda$.

PROOF. By (1.7) we may assume that G is split with maximal torus T. Let $A(\mathfrak{g})^G = k[a_1, \ldots, a_r]$ where $a_1, \ldots, a_r$ are algebraically independent and a_i is homogeneous of degree $d_{r+1-i} = 1 + \lambda_i$, cf. [7, Theorem 3]. We have $\mathcal{O}_{N(G),x} = \mathcal{O}_{\mathfrak{g},x}/\langle \mathrm{a} \rangle$. Since $\mathcal{O}_{\mathfrak{g},x}$ is flat over $A(\mathfrak{g})^G$ the sequence a is $\mathcal{O}_{\mathfrak{g},x}$-regular. By (1.6) (a) this implies that $\mathrm{ord}(x, N(G)/k) \geqslant \lambda$. Let $\mathfrak{p} \in \mathrm{Spec}(k)$ be the image of x. By (1.5) we may replace k by the residue field $\mathrm{k}(\mathfrak{p})$. Now the assertion follows from the example in (1.4).

REMARK. If k is noetherian regular the multiplicity of the local ring $\mathcal{O}_{N(G),x}$ is equal to $\Pi_{i=1}^r d_i$, i.e. the order of the Weyl group. This is proved in [13, p. 55] using the methods of [18]. Compare [16, p. 386].

3. In the classical Lie-algebras.

(3.1) We fix a free k-module F of rank n. The scheme $\mathrm{End}(F)$ is defined by $\mathrm{End}(F)(R) := \mathrm{End}_R(F \otimes_k R)$, cf. [11, I 9]. The group scheme $Gl(F)$ (resp. $Sl(F)$) is the open (resp. closed) subscheme of $\mathrm{End}(F)$ where the function det $\in A(\mathrm{End}(F))$ is invertible (resp. where $\det = 1$). $Gl(F)$ and $Sl(F)$ are reductive group schemes over k of type A_{n-1}, cf. [6] and [8]. $\mathrm{End}(F)$ is identified with $\mathrm{Lie}(Gl(F))$ by $x \leftrightarrow 1 + \delta x$ where $x \in \mathrm{End}(F)(R)$, see (2.3) or [8, II 4]. Now $\mathrm{Lie}(Sl(F))$ consists of the endomorphisms with zero trace.

Assume $1/2 \in k$. Let ϵ be 0 or 1. An *ϵ-form* ϕ on F is a nondegenerate bilinear form ϕ: $F \times F \longrightarrow k$ which is symmetric if $\epsilon = 0$, alternating if $\epsilon = 1$. By "nondegenerate" we mean that the mapping $F \longrightarrow F^{\vee}$ defined by $f \mapsto \phi(f, -)$ is bijective. Let ϕ be an ϵ-form. The subgroup functor $G'(F, \phi)$ of $Gl(F)$ is defined by $x \in G'(F, \phi)(R)$ if and only if

$$\phi(xf, xg) = \phi(f, g) \qquad (f, g \in F \otimes R).$$

We define $G(F, \phi) := G'(F, \phi) \cap Sl(F)$. If $\epsilon = 0$ then $G(F, \phi)$ is the special orthogonal group scheme. If $\epsilon = 1$ then $G(F, \phi) = G'(F, \phi)$; it is the symplectic group scheme. Put $l := [\frac{1}{2}n]$ and $\zeta := n - 2l$. So ζ is 0 or 1. If $\epsilon = 1$ then $\zeta = 0$. Now $G(F, \phi)$ is a semisimple group scheme of type B_l if $\epsilon = 0, \zeta = 1$, of type C_l if $\epsilon = 1, \zeta = 0$, of type D_l if $\epsilon = \zeta = 0$, cf. [6] and [8]. The common Lie-algebra of $G(F, \phi)$ and $G'(F, \phi)$ is denoted by $\mathfrak{g}(F, \phi)$. For $x \in \operatorname{End}(F)(R)$ we have $x \in \mathfrak{g}(F, \phi)(R)$ if and only if

$$\phi(xf, g) + \phi(f, xg) = 0 \qquad (f, g \in F \otimes R).$$

CONVENTION. In the rest of this paper we consider two cases.

Case I. $G := G' := Gl(F), l := n$.

Case II. (ϵ, ζ) where $\epsilon, \zeta \in \{0, 1\}, \epsilon + \zeta \leqslant 1$: $1/2 \in k$, ϕ is an ϵ-form on F, $G := G(F, \phi), G' := G'(F, \phi), n = 2l + \zeta$.

In both cases l is the reductive rank of G. We put $\mathfrak{g} := \operatorname{Lie}(G)$. While considering Case II it is convenient to label concepts introduced for Case I with the index I, e.g. $\mathfrak{g} \subset \mathfrak{g}_I = \operatorname{End}(F)$.

(3.2) LEMMA. *Case* II. *Let ϕ_1 be another ϵ-form on F. Then there is a faithfully flat étale k-algebra R such that ϕ_1 and ϕ induce equivalent forms on $F \otimes R$.*

PROOF. By [15, pp. 34, 35] the scheme $\operatorname{Isom}(\phi_1, \phi)$ is smooth over k. If K is an algebraically closed field over k then $\operatorname{Isom}(\phi_1, \phi)(K) \neq \emptyset$. Hence by [12, IV 17.16.3] there is a faithfully flat étale k-algebra R with $\operatorname{Isom}(\phi_1, \phi)(R) \neq \emptyset$.

(3.3) DEFINITION. In Case I, $z \in \mathfrak{g}(R)$ is called a *standard nilpotent* with *base-data* (f, λ) if $\mathrm{f} = (f_1, \ldots, f_r)$ is a sequence in $F \otimes R$ and λ is a partition, such that $\lambda^1 = r$, that the set $\{z^a f_i\}$, where $1 \leqslant i \leqslant r$ and $0 \leqslant a < \lambda_i$, is a basis of $F \otimes R$ and that $z^a f_i = 0$ if $a \geqslant \lambda_i$.

In Case II, $z \in \mathfrak{g}(R)$ is called a *standard nilpotent* with *base-data* $(\mathrm{f}, \lambda, \beta, \alpha)$ if $z \in \mathfrak{g}_I(R)$ is a standard nilpotent with base-data (f, λ), β is a permutation of $\{1, \ldots, r\}$ where $r = \lambda^1$, and α: $\{1, \ldots, r\} \longrightarrow R$ is a mapping such that

$$(1) \qquad \begin{cases} \phi(z^a f_i, z^b f_j) = (-1)^a \alpha(i) & \text{if } j = \beta i \text{ and } a + b + 1 = \lambda_i, \\ \phi(z^a f_i, z^b f_j) = 0 & \text{otherwise.} \end{cases}$$

REMARK. Clearly $|\lambda| = n$. In Case II the assumptions imply

$$(2)\qquad \alpha(i)^{-1} \in R,\quad \beta^2 = \mathrm{id},\quad \lambda_{\beta i} = \lambda_i,\quad \alpha(\beta i) = (-1)^{\lambda_i - 1 + \epsilon}\alpha(i).$$

(3.4) The set P_ϵ is defined as the subset of P consisting of the partitions λ such that for any $m \geqslant 1$ with $m \equiv \epsilon$ (2) the number of indices i with $\lambda_i = m$ is even. These partitions are called orthogonal, resp. symplectic; in [10, p. 556]. We define $P(n)$ as the set of partitions λ with $|\lambda| = n$, and $P_\epsilon(n) := P_\epsilon \cap P(n)$. We write $P_{(\epsilon)}$ to denote P in Case I and P_ϵ in Case II. So in (3.3) we have $\lambda \in P_{(\epsilon)}(n)$.

(3.5) If $x \in \mathfrak{g}$ is nilpotent, cf. (2.3), then the section $x \in \mathfrak{g}(\mathrm{k}(x))$ is a standard nilpotent by [20, IV]. Let $\lambda \in P(n)$. We define $\mathfrak{O}(\lambda)$ as the set of points $x \in \mathfrak{g}$ such that the section x is a standard nilpotent with partition λ. In case II we have $\mathfrak{O}(\lambda) = \mathfrak{O}_I(\lambda) \cap \mathfrak{g}$, and $\mathfrak{O}(\lambda) \neq \varnothing$ if and only if $\lambda \in P_\epsilon(n)$.

Let k be a field and $x \in \mathfrak{O}(\lambda)$. By [20, IV] we have $\mathfrak{O}(\lambda) = G'x$, and $\mathfrak{O}(\lambda) \neq Gx$ if and only if we are in the *very-even case*: Case II (0, 0) with λ_i even for all i.

(3.6) LEMMA. *Case* I. *If* $\lambda \in P(n)$, *there is a standard nilpotent element* $z \in \mathfrak{g}(k)$ *with partition* λ.

Case II. *If* λ, β *and* α *satisfy the conditions* (3.3)(2), *then there is an* ϵ-*form* ϕ_1 *on* F *and a standard nilpotent element* $z \in \mathfrak{g}(F, \phi_1)(k)$ *with base-data* $(\mathbf{f}, \lambda, \beta, \alpha)$ *for some sequence* $\mathbf{f}$ *in* F.

PROOF. Case I is trivial.

Case II. Choose a standard nilpotent $z \in \mathfrak{g}_I(k)$ with base-data $(\mathbf{f}, \lambda)$. Let $\phi_1\colon F \otimes F \longrightarrow k$ be the bilinear form defined by (3.3)(1). One verifies that ϕ_1 is an ϵ-form on F with $z \in \mathfrak{g}(F, \phi_1)(k)$.

(3.7) *The standard cross section.* Let $z \in \mathfrak{g}(k)$ be a standard nilpotent element with base-data $(\mathbf{f}, \lambda)$, resp. $(\mathbf{f}, \lambda, \beta, \alpha)$. Below we construct a linear subscheme $L \subset \mathfrak{g}$ such that $\mathfrak{g}(R) = [\mathfrak{g}(R), z] \oplus L(R)$ for any k-algebra R. This implies that *the subscheme* $z + L \subset \mathfrak{g}$ *is a cross section for the adjoint action of* G *in all points of the section* z, cf. (2.1). In fact the tangent morphism of $\mathrm{Ad}\colon G \times (z + L) \longrightarrow \mathfrak{g}$ at the section (e, z) is the surjective morphism $\mathfrak{g} \oplus L \longrightarrow \mathfrak{g}$ given by $(x, y) \longmapsto [x, z] + y$. So smoothness of Ad at (e, z) follows from [12, IV 17.11.1].

Let Ψ be the set of pairs (i, a) with $0 \leqslant a < \lambda_i$. Put $f(i, a) := z^a f_i$. Then $\{f(\psi) | \psi \in \Psi\}$ is a basis of F. Let $\{u(\psi)\}$ be the dual basis of $F^\vee$. This means that $\{u(\psi)\}$ is the basis of $F^\vee = \mathrm{Hom}(F, k)$ with

$$\langle u(\psi), f(\psi') \rangle = \delta_{\psi, \psi'} \quad \text{(Kronecker delta).}$$

The coordinates $\xi(\psi; \psi')$ on $\mathfrak{g}_I$ are defined by $\xi(\psi; \psi')(x) = \langle u(\psi), xf(\psi') \rangle$.

Clearly $\{\xi(\psi;\psi')|\psi,\psi'\in\Psi\}$ is a basis of $\mathfrak{g}_I(k)$. Let $\{e(\psi;\psi')\}$ be the dual basis of $\mathfrak{g}_I(k)$. We have

$$e(\psi;\psi')f(\psi'')=\delta_{\psi',\psi''}f(\psi),$$

$$[e(i,a;j,b),z]=e(i,a;j,b-1)-e(i,a+1;j,b)$$

where $e(i,a;j,b)=0$ if $a\geqslant\lambda_i$ or $b<0$. In Case I let $\mathfrak{g}_{ij}$, L_{ij} and L be the linear subschemes of $\mathfrak{g}$ defined by

$$\mathfrak{g}_{ij}(R):=\sum_{a,b}Re(i,a;j,b),$$

$$L_{ij}(R):=\sum Re(i,a;j,\lambda_j-1),\quad 0\leqslant a<\min(\lambda_i,\lambda_j),$$

$$L(R):=\sum_{i,j}L_{ij}(R).$$

Then we have $\mathfrak{g}_{ij}=[\mathfrak{g}_{ij},z]\oplus L_{ij}$ and $\mathfrak{g}=[\mathfrak{g},z]\oplus L$.

Case II. The coordinates $\eta(\psi;\psi')$ on $\mathfrak{g}$ are defined by $\eta(\psi;\psi')(x)=\phi(f(\psi),xf(\psi'))$. Since $\eta(\psi;\psi')=(-1)^{1+\epsilon}\eta(\psi';\psi)$ we have a basis of $\mathfrak{g}(k)^\vee$ consisting of the $\eta(i,a;j,b)$ with $i<j$, or $i=j$ and $a<b+\epsilon$. Let $y(\psi;\psi')$ be the dual basis of $\mathfrak{g}(k)$. One shows that

$$[y(i,a;j,b),z]=y(i,a;j,b-1)+y(i,a-1;j,b)$$

$$\text{if } i<j, \text{ or } i=j \text{ and } a<b-1,$$

$$[y(i,a;i,a+1),z]=y(i,a-1;i,a+1)+2\epsilon y(i,a;i,a),$$

$$[y(i,a;i,a),z]=y(i,a-1;i,a)\quad\text{if } \epsilon=1,$$

where $y(\psi;\psi')=0$ if not yet defined. For $i\leqslant j$ let $\mathfrak{g}_{ij}$, L_{ij} and L be the linear subschemes of $\mathfrak{g}$ defined by

$$\mathfrak{g}_{ij}(R):=\sum_{a,b}Ry(i,a;j,b),$$

$$L_{ij}(R):=\sum_{b}Ry(i,\lambda_i-1;j,b)\quad\text{if } i<j,$$

$$L_{ii}(R):=\sum Ry(i,\lambda_i-2+\epsilon-a;i,\lambda_i-1-a),\ 0\leqslant a\leqslant \tfrac{1}{2}(\lambda_i-2+\epsilon),$$

$$L(R):=\sum_{i\leqslant j}L_{ij}(R).$$

Then we have $\mathfrak{g}_{ij}=[\mathfrak{g}_{ij},z]\oplus L_{ij}$ and $\mathfrak{g}=[\mathfrak{g},z]\oplus L$.

Let $F^\vee$ be identified with F in such a way that $\langle u,f\rangle=\phi(u,f)$. Putting $|i|_a=(-1)^a\alpha(i)^{-1}$, we get the following glossary:

$$|i|_a f(i, a) = u(\beta i, \lambda_i - 1 - a),$$

$$|i|_a \eta(i, a; j, b) = \xi(\beta i, \lambda_i - 1 - a; j, b)|\mathfrak{g},$$

$$y(i, a; j, b) = |i|_a e(\beta i, \lambda_i - 1 - a; j, b) - (-1)^\epsilon |j|_b e(\beta j, \lambda_j - 1 - b; i, a)$$

$$\text{if } i < j, \text{ or } i = j \text{ and } a < b,$$

$$y(i, a; i, a) = |i|_a e(\beta i, \lambda_i - 1 - a; i, a) \quad \text{if } \epsilon = 1,$$

$$y(i, a; j, b) = 0 \quad \text{otherwise.}$$

REMARK. In Case I our $z + L$ is one of the cross sections of Arnold [1].

(3.8) An elementary calculation shows that $L(k)$ is a free k-module of rank $l + \gamma_{(\epsilon)}(\lambda)$ if we write $\gamma_{(\epsilon)}(\lambda) := \gamma(\lambda)$ in Case I and $\gamma_{(\epsilon)}(\lambda) := \gamma_\epsilon(\lambda)$ in Case II where

$$\gamma(\lambda) := 2\sum (i-1)\lambda_i \quad \text{if } \lambda \in P(n),$$

$$\gamma_\epsilon(\lambda) := \sum (i-1)\lambda_i + (2\epsilon - 1)[\tfrac{1}{2}\, \#\{i \mid \lambda_i \equiv 1 \ (2)\}] \quad \text{if } \lambda \in P_\epsilon(n).$$

Now the centralizer of z in $\mathfrak{g}(k)$ is also a free k-module of rank $l + \gamma_{(\epsilon)}(\lambda)$. By [20, I 5.6] we have the following:

COROLLARY. *Assume that k is a field.*

(a) *If* $x \in \mathfrak{O}(\lambda)$ *then* $\dim(Gx) = \dim(\mathfrak{g}) - l - \gamma_{(\epsilon)}(\lambda)$.

(b) *There is a unique nilpotent orbit* C_{reg} *of maximal dimension.*

$C_{\text{reg}} = \mathfrak{O}(\nu)$ *where* $\nu_* = (n)$ *in the Cases* I *and* II $(\epsilon, 1-\epsilon)$ *and* $\nu_* = (\eta - 1, 1)$ *in Case* II $(0, 0)$. *We have* $\dim(C_{\text{reg}}) = \dim(\mathfrak{g}) - l$. *If C is another nilpotent orbit in* $\mathfrak{g}$ *then* $\dim(C) \leqslant \dim(\mathfrak{g}) - l - 2$.

See also [1], [20, IV 2.28] and [21, p. 136].

(3.9) The mapping Σ: $P(n) \longrightarrow P$ is defined by $(\Sigma\lambda)^m := \Sigma_{i>m}\lambda_i$ $(m \in \mathbb{N})$. As the corresponding propositions in [10, p. 567] are false, we shall prove the following:

PROPOSITION. *Let* $\lambda, \mu \in P_{(\epsilon)}(n)$ *be such that*

$$\{\mu\} = \{\nu \in P_{(\epsilon)}(n) \mid \Sigma\lambda > \Sigma\nu \geqslant \Sigma\mu\}.$$

Then there are $\rho, \sigma, \tau \in P_{(\epsilon)}$ *with* $\lambda = \rho + \sigma$, $\mu = \rho + \tau$ *and* σ, τ *as described in the following table.*

Case		σ_*	τ_*	Restrictions	
I		(p, q)	$(p+1, q-1)$	$p \geqslant q \geqslant 1$	
II	(a)	(p, p)	$(p+1, p-1)$	$p \geqslant 1$ and $p \equiv \epsilon\ (2)$	
	(b_1)	(p, q)	$(p+2, q-2)$	$p \geqslant q \geqslant 2$	and $p \equiv q \not\equiv \epsilon\ (2)$
	(b_2)	(p, p, q)	$(p+1, p+1, q-2)$	$p \geqslant q \geqslant 2$	and $p \equiv q \not\equiv \epsilon\ (2)$
	(b_3)	(p, q, q)	$(p+2, q-1, q-1)$	$p \geqslant q \geqslant 1$	and $p \equiv q \not\equiv \epsilon\ (2)$
	(b_4)	(p, p, q, q)	$(p+1, p+1, q-1, q-1)$	$p \geqslant q \geqslant 1$	and $p \equiv q \not\equiv \epsilon\ (2)$

PROOF. See (1.1) for the addition of partitions. Case I may be left to the reader. Case II. It is easy to see that we may assume disjointness: if $\lambda_i = \mu_j$ then $\lambda_i = 0$. Now we have to prove $\lambda = \sigma, \mu = \tau$ as in the table.

(a) Assume that there is a minimal $l \in \mathrm{N}$ with λ_l with $\lambda_l \neq 0$ and $\lambda_l \equiv \epsilon\ (2)$. There is a maximal $m \in \mathrm{N}$ with $\lambda_m = \lambda_l$. Define $\nu \in P_\epsilon(n)$ by $\nu_l = \lambda_l + 1, \nu_m = \lambda_m - 1$ and $\nu_i = \lambda_i$ otherwise. Clearly $\Sigma\lambda > \Sigma\nu$. Using disjointness one proves $\Sigma\nu \geqslant \Sigma\mu$, so that $\nu = \mu$ and, again by disjointness, we are in case (a).

(b) Now $\lambda_i \not\equiv \epsilon\ (2)$ whenever $\lambda_i > 0$. By disjointness there is an $m \in \mathrm{N}$ with $\mu_m > \lambda_1 > \mu_{m+1}$. It is easy to see that we can define $\nu \in P_\epsilon(n)$ satisfying $\Sigma\nu > \Sigma\mu$ as follows:

If $\mu_m \not\equiv \epsilon\ (2)$, then $\nu_m = \mu_m - 2$ and $\nu_i = \mu_i$ if $i < m$;
if $\mu_m \equiv \epsilon\ (2)$, then $\nu_{m-1} = \nu_m = \mu_m - 1$ and $\nu_i = \mu_i$ if $i < m - 1$;
if $\mu_{m+1} \not\equiv \epsilon\ (2)$, then $\nu_{m+1} = \mu_{m+1} + 2$ and $\nu_i = \mu_i$ if $i > m + 1$;
if $\mu_{m+1} \equiv \epsilon\ (2)$, then $\nu_{m+1} = \nu_{m+2} = \mu_{m+1} + 1$ and $\nu_i = \mu_i$ if $i > m + 2$.

One proves that $\Sigma\lambda \geqslant \Sigma\nu$, so that $\lambda = \nu$ and we are in one of the four cases (b).

(3.10) THEOREM. *Let k be a field. Consider $z \in \mathfrak{O}(\lambda)$ and $x \in \mathfrak{O}(\mu)$. We have $z \in \overline{Gx} - Gx$ if and only if $\Sigma\lambda > \Sigma\mu$.*

REMARK. This theorem is due to Gerstenhaber, see [9, p. 327] and [10, pp. 567–569]. His proof for Case II is incomplete, see (3.9). Our proof seems to be more explicit.

PROOF. We may assume that z and x are rational points. So z is a standard nilpotent in $\mathfrak{g}(k)$ with partition λ. If $i \in \mathrm{N}$ then the endomorphism z^i of F has rank $(\Sigma D\lambda)^i$, see (1.1) for the definition of D.

Assume that $z \in \overline{Gx} - Gx$. The rank of z^i is less than or equal to the rank of x^i. This implies $\Sigma D\lambda \leqslant \Sigma D\mu$ and hence $\Sigma\lambda \geqslant \Sigma\mu$ by [9, p. 327]. As $\lambda \neq \mu$ it is easy to see that $\Sigma\lambda > \Sigma\mu$.

Assume that $\Sigma\lambda > \Sigma\mu$. We have to prove that $z \in \overline{Gx}$. We may assume that λ and μ are as in (3.9). So λ and μ are not both very-even, cf. (3.5), and it suffices to prove that $z \in \overline{\mathfrak{O}(\mu)}$. Using the notations of (3.7) we shall construct $y \in \mathfrak{g}(k)$ and a sequence $\mathrm{f}(t)$ $(t \in k)$ in such a way that $z(t) = z + ty \in$

$\mathfrak{g}(k)$ is a standard nilpotent in $\mathfrak{g}_I(k)$ with base-data $(\mathrm{f}(t), \mu)$ if $t \neq 0$. This will prove $z \in \overline{\mathfrak{O}(\mu)}$.

Using a direct sum decomposition we may assume $\rho = 0, \lambda = \sigma, \mu = \tau$; cf. (3.9).

Case I. We have $\lambda_* = (p, q)$ and $\mu_* = (p + 1, q - 1)$. Let $((f_1, f_2), \lambda)$ be base-data for z. Put $y := e(1, q - 1; 2, q - 1)$. Put $f_1(t) := f_2$ and, if $q > 1, f_2(t) := tf_1 - zf_2$. We have

$$0 \leqslant a \leqslant q - 1 \Rightarrow z(t)^a f_1(t) = z^a f_2,$$

$$q \leqslant a \leqslant p \Rightarrow z(t)^a f_1(t) = tz^{a-1} f_1,$$

$$0 \leqslant a \leqslant q - 2 \Rightarrow z(t)^a f_2(t) = tz^a f_1 - z^{a+1} f_2,$$

$$z(t)^{p+1} f_1(t) = 0 \quad \text{and} \quad z(t)^{q-1} f_2(t) = 0.$$

This implies that $z(t) \in \mathfrak{O}(\mu)$ if $t \neq 0$.

Case II. Of the five possibilities, cf. (3.9), we only treat (b_3) and (b_4) with $q \geqslant 2$. The other cases are easier, see [13, (4.3.7)], and already settled in [10, pp. 568, 569]. We choose convenient base-data $((f_1, \ldots, f_r), \lambda, \beta, \alpha)$ for z. The verifications are left to the reader.

(b_3) $\lambda_* = (p, q, q), p \equiv q \not\equiv \epsilon\ (2), r = 3, \beta = \mathrm{id}, \mu_* = (p + 2, q - 1, q - 1)$. Choose

$$\begin{aligned} y &:= y(1, p - 1; 2, 0) + y(1, p - 1; 3, 0) \\ &= e(1, 0; 2, 0) + e(1, 0; 3, 0) + e(2, q - 1; 1, p - 1) - e(3, q - 1; 1, p - 1), \end{aligned}$$

$f_1(t) := f_2, f_2(t) := zf_2$ and $f_3(t) := z^{p-q+1} f_1 - tf_2 + tf_3$.

(b_4) $\lambda_* = (p, p, q, q), p \equiv q \not\equiv \epsilon\ (2), r = 4, \beta = \mathrm{id}, \mu_* = (p + 1, p + 1, q - 1, q - 1)$. Choose

$$\begin{aligned} y := {} & y(1, p - 1; 3, 0) + y(1, p - 1; 4, 0) \\ & + y(2, p - 1; 3, 0) + y(2, p - 1; 4, 0) \\ = {} & e(1, 0; 3, 0) + e(1, 0; 4, 0) + e(2, 0; 3, 0) + e(2, 0; 4, 0) \\ & + e(3, q - 1; 1, p - 1) - e(4, q - 1; 1, p - 1) \\ & - e(3, q - 1; 2, p - 1) + e(4, q - 1; 2, p - 1), \end{aligned}$$

$f_1(t) := f_1, f_2(t) := f_3, f_3(t) := zf_3$ and $f_4(t) := z^{p-q+1} f_1 - tf_3 + tf_4$.

4. The classical nilpotent scheme, singularities.

(4.1) The *symmetrical polynomials* $\sigma_1, \ldots, \sigma_n \in A(\mathrm{End}(F))$ are defined by the equation

$$\det(x + T \cdot \mathrm{id}) = T^n + \sum_{m=1}^{n} T^{n-m} \sigma_m(x)$$

in $R[T]$ where R is a k-algebra and $x \in \mathrm{End}(F)(R)$. They are invariant under the adjoint action of $Gl(F)$ on $\mathrm{End}(F)$. Let $X = (x_{ij})$ be the matrix of x with respect to some basis $f_1, \ldots, f_n$ of F. Then

$$\sigma_m(x) = \sum \det(x_{ij})_{i,j \in I}$$

where the summation is over all subsets I of $\{1, \ldots, n\}$ with $\# I = m$.

Case II. Clearly $\sigma_m | \mathfrak{g} \in A(\mathfrak{g})^G$. Let Φ be the matrix $\phi(f_i, f_j)$. We have $x \in \mathfrak{g}(R)$ if and only if ${}^tX = -\Phi X \Phi^{-1}$. This implies that $\sigma_m | \mathfrak{g} = 0$ if m is odd. Assume $\epsilon = \zeta = 0$. We define $\tau_l \in A(\mathfrak{g})$ by $\tau_l(x) := \mathrm{Pf}(\Phi X)$, where Pf denotes the Pfaffian, cf. [3, §5, no. 2]. Using loc. cit. one proves that $\tau_l^2 = \det(\Phi)\sigma_n$ and that $\tau_l \in A(\mathfrak{g})^G$.

We define the sequence $\mathrm{a} = (a_1, \ldots, a_l)$ in $A(\mathfrak{g})$ as follows. In Case I we put $a_i := \sigma_i$. In Case II $(\epsilon, 1-\epsilon)$ we put $a_i := \sigma_{2i}$. In Case II (0, 0) we put $a_i := \sigma_{2i}$ if $i < l$, and $a_l := \tau_l$.

THEOREM. (a) *$A(\mathfrak{g})^G$ is the free polynomial ring $k[a_1, \ldots, a_l]$.*

(b) *The sequence* a *is $A(\mathfrak{g})$-regular (in any order).*

(c) *$N(G) = \mathrm{Spec}(A(\mathfrak{g})/\langle \mathrm{a} \rangle)$, it is flat over k.*

(d) *$N(G)$ is smooth over k in the points of $\mathfrak{O}(\nu)$ where ν is, cf.* (3.8)(b).

(e) *If k is a normal ring then $N(G)$ is a normal scheme.*

PROOF. (a) Let u: $k[T_1, \ldots, T_l] \longrightarrow A(\mathfrak{g})^G$ be defined by $T_i \longmapsto a_i$. We have to prove that u is bijective. Replacing k by a faithfully flat k-algebra (cf. (3.6) and (3.2)), we may assume the existence of a standard nilpotent $z \in \mathfrak{g}(k)$ with partition ν, cf. (3.8)(b). Let $z + L$ be the cross section of (3.7). By (2.2) the morphism v: $A(\mathfrak{g})^G \longrightarrow A(z + L)$ is injective. Case by case one shows that $v \circ u$ is bijective, so that u is bijective.

(b) and (c). By [7], Theorem (2.6) applies. So $A(\mathfrak{g})$ is flat over $A(\mathfrak{g})^G$. So we have (b) and (c).

(d) We may use the cross section of (a). Now $(z + L) \cap N(G)$ is a cross section at z for the action of G on $N(G)$, and the assertion follows from $(z + L) \cap N(G) \cong \mathrm{Spec}(k)$.

(e) By (c) and [12, IV 6.14.1] we may assume that k is a field. Now $N(G)$ is nonsingular in codimension one, by (d) and (3.8)(b). So $N(G)$ is normal by Serre's criterion [12, IV 5.8.6].

REMARKS. (i) There are other ways to prove the theorem, either avoiding (3.7) or avoiding (2.6) and [7]. (ii) It can be shown that $N(Sl(F))$ exists and is equal to $N(Gl(F))$, where k is arbitrary. Here (2.6) does not apply.

(4.2) If $\lambda \in P(n)$, the partition $\Sigma\lambda$ is defined in (3.9). Case II $(\epsilon, 1-\epsilon)$. If $\lambda \in P_\epsilon(n)$ where $n = 2l + 1 - \epsilon$, then we define the partition $\Sigma_\epsilon\lambda$ by $(\Sigma_\epsilon\lambda)_i := (\Sigma\lambda)_{2i-\epsilon}$. Case II (0, 0). If $\lambda \in P_0(n)$ where $n = 2l$, then we define $\Sigma_0\lambda$

$:= \theta + \nu$ where $\theta, \nu \in P$ are given by $\theta_i := (\Sigma\lambda)_{2i+1}$ and $\nu_* := (\frac{1}{2}\lambda^1 - 1)$. Note: in the last case λ^1 is even and $(\Sigma\lambda)_1 = \lambda^1 - 1$. We write $\Sigma_{(\epsilon)}$ to denote Σ in Case I and Σ_ϵ in Case II. $\Sigma_{(1)}$ means that $\epsilon = 0$ is excluded in Case II.

THEOREM. *Consider* $x \in \mathfrak{O}(\lambda)$. *Then* $\operatorname{ord}(x, N(G)/k) = \Sigma_{(1)}\lambda$ *in Cases* I *and* II $(1, 0)$, *and* $\operatorname{ord}(x, N(G)/k) \geq \Sigma_0\lambda$ *in Case* II $(0, \zeta)$.

PROOF. (1) By (1.7) we may replace $(N(G), x)$ by a smoothly equivalent pointed scheme. So by (3.6) and (3.2) we may assume the existence of a standard nilpotent $z \in \mathfrak{g}(k)$ with partition λ. By (2.1) (a) we may assume that $x = z(\mathfrak{p})$ for some $\mathfrak{p} \in \operatorname{Spec}(k)$. Put $A := \mathcal{O}_{\mathfrak{g},x}$ and $B := \mathcal{O}_{N(G),x}$. We have $B = A/\langle \mathrm{a} \rangle$ where a is the A-regular sequence of (4.1), or rather its image in A.

(2) Let J be the ideal in $A(\mathfrak{g})$ corresponding to the section z. So x corresponds to the prime ideal $J + \mathfrak{p}A(\mathfrak{g})$. We claim

(a) If $1 \leq i \leq n$ and $m := 1 + (\Sigma\lambda)_{n+1-i}$ then $\sigma_i \in J^m$.

(b) In Case II (0, 0) we have $\tau_l \in J^m$ where $m := \frac{1}{2}\lambda^1$.

PROOF OF (a). It suffices to consider Case I. Let (f, λ) be base-data for z. Using the notation of (3.7) we define

$$\sigma_P := \det \xi(\psi; \psi')_{\psi, \psi' \in P}$$

if $\varnothing \neq P \subset \Psi$. So $\sigma_i = \Sigma\sigma_P$ where the summation is over all P with $\#P = i$. If $\xi(\psi; \psi') \notin J$ then we have $\psi' = (j, a)$, $\psi = (j, a + 1)$ for some j and a. Consider P with $\#P = i$. If π is a permutation of P then one verifies that

$$\#\{(j, a) \in P | \pi(j, a) \neq (j, a + 1)\} \geq 1 + (\Sigma\lambda)_{n+1-i} = m.$$

This implies $\sigma_P \in J^m$, proving (a).

PROOF OF (b). We may assume that k is reduced. Now the assertion follows from $\tau_l^2 = \det(\Phi)\sigma_n \in J^{2m}$, cf. (a).

(3) By (1.6)(a) it follows from (2)(a), (b) that $\operatorname{ord}(B/k) \geq \Sigma_{(\epsilon)}\lambda$. This proves the theorem in Case II $(0, \zeta)$. In the rest of the proof Case II $(0, \zeta)$ is excluded. It suffices to prove

$$(*) \qquad \operatorname{ord}(B/k) \leq \Sigma_{(1)}\lambda.$$

By (1.5) and (4.1)(c) we may replace k by an algebraic closure of the field $\mathrm{k}(\mathfrak{p})$. So henceforth k is an algebraically closed field. Now x and z may be identified. Let (f, λ), resp. $(\mathrm{f}, \lambda, \beta, \alpha)$ be its base-data.

(4) We prove (3) $(*)$ by induction on $n = |\lambda|$. The cases with $n \leq 1$ are trivial. So assume $n \geq 2$. Put $r := \lambda^1$. Let ρ be the partition with $\rho_* = (r - 1)$. The partition μ is defined as follows.

Case I. $\mu_r := \lambda_r - 1$, $\mu_i := \lambda_i$ if $i \neq r$.

Case II. If λ_r is even then $\mu_r := \lambda_r - 2$ and $\mu_i := \lambda_i$ otherwise. If λ_r is odd so that $\lambda_{r-1} = \lambda_r$, then $\mu_{r-1} := \mu_r := \lambda_r - 1$ and $\mu_i := \lambda_i$ otherwise.

One verifies that $\mu \in P_{(1)}$ and that $\Sigma_{(1)}\lambda = \rho + \Sigma_{(1)}\mu$.

We have $F = \Sigma kf(\psi)$, $\psi \in \Psi$, cf. (3.7). Let P be the subset of Ψ containing $(r, 0)$ and in Case II also $(\beta r, \lambda_r - 1)$. Put $F' := \Sigma kf(\psi)$, $\psi \notin P$, and $F'' := \Sigma kf(\psi)$, $\psi \in P$. Clearly $F = F' \oplus F''$. In Case II the form $\phi' := \phi|F'$ is nondegenerate and hence a 1-form on F'. We put $G' := Gl(F')$ in Case I and $G' := G(F', \phi')$ in Case II. So the convention (3.1) concerning G' is not applied here. We put $\mathfrak{g}' := \mathrm{Lie}(G')$, etc.

Let

$$\begin{pmatrix} x' & x_2 \\ x_1 & x_3 \end{pmatrix}$$

be the matrix of x with respect to the decomposition $F = F' \oplus F''$. Now x' is a standard nilpotent in $\mathfrak{g}'(k)$ with partition μ. Consider the ring

$$B' := \mathcal{O}_{N(G'),x'} = \mathcal{O}_{\mathfrak{g}',x'}/\langle(\sigma_i')_{i<n}\rangle.$$

By induction we have $\mathrm{ord}(B'/k) \leqslant \Sigma_{(1)}\mu$. One verifies that

$$w \longmapsto \begin{pmatrix} w & x_2 \\ x_1 & x_3 \end{pmatrix}$$

defines a regular immersion u: $\mathfrak{g}' \longrightarrow \mathfrak{g}$ such that $u(x') = x$, $u^0(\sigma_n) = 0$ and $u^0(\sigma_i) = \sigma_i'$ if $i < n$, where u^0: $A(\mathfrak{g}) \longrightarrow A(\mathfrak{g}')$ is the comorphism. Put $R := A/\langle(\sigma_i)_{i<n}\rangle$ so that $B = R/\langle f\rangle$ where f is the image of σ_n in R. Now there is an R-regular sequence x in R such that $B' \cong R/\langle \mathrm{x}\rangle$ and $f \in \langle \mathrm{x}\rangle$. By (1.6)(b) this implies

$$\mathrm{ord}(B/k) \leqslant \rho + \mathrm{ord}(B'/k) \leqslant \rho + \Sigma_{(1)}\mu = \Sigma_{(1)}\lambda$$

provided that $f \notin m_R^{r+1}$. So in order to prove the theorem it suffices to prove that

$$(*) \qquad \sigma_\Psi \notin \langle(\sigma_P)_{P\neq\Psi}\rangle + \mathfrak{m}_A^{r+1}$$

where we have used the notation of (2).

(5) In Case II we normalize the base-data of x as follows: $\beta i \neq i$ if and only if λ_i is odd; $|\beta i - i| \leqslant 1$ for all i; if $i \geqslant \beta i$ then $\alpha(i) = 1$. Now $i \leqslant \beta i$ implies $\alpha(i) = (-1)^{\lambda_i}$. With the notation of (3.7) we define a linear subvariety M of $\mathfrak{g}$.

Case I. $M := \Sigma ke(i, 0; j, \lambda_j - 1)$ $(1 \leqslant i, j \leqslant r)$.

Case II. $M := \Sigma ky(i, \lambda_i - 1; j, \lambda_j - 1)$ where the summation is over all pairs (i, j) such that $i = j$ or $i \leqslant \beta i < j \leqslant \beta j$. So in this case $M \subset M_I$.

The ring $A(x + M)$ is considered as a graded k-algebra such that x corresponds to the augmentation ideal. The functions $\sigma_P|x + M$ are homogeneous,

$\sigma_\Psi | x + M$ is homogeneous of degree r. So it suffices to prove

$$(*) \qquad \sigma_\Psi | x + M \notin \langle\langle \sigma_P | x + M)_{P \neq \Psi}\rangle.$$

(6) *Case* I. It is easy to see that $x + M$ has a subvariety $x_1 + M_1$ such that $\sigma_P | x_1 + M_1 \neq 0$ if and only if $P = \Psi$. This proves (5)(*) and the theorem.

Case II. Consider the subvariety $x_1 + M_1$ of $x + M$ where

$$x_1 := x + \sum_{i > \beta i} y(i, \lambda_i - 1; i, \lambda_i - 1),$$

$$M_1 := \sum k y(i, \lambda_i - 1; j, \lambda_j - 1) \ \ (i \leqslant \beta i, \ i \leqslant j \leqslant \beta j).$$

Now x_1 is a standard nilpotent in $\mathfrak{g}_I(k)$ with base-data $(\mathfrak{f}', \lambda')$ such that

$$M_1 = \sum_{i<j} k(e'(i, 0; j, \lambda_j' - 1) + e'(j, 0; i, \lambda_i' - 1))$$

with respect to the new base-data. In order to prove (5)(*) and hence the theorem, it suffices to show that

$$\sigma_\Psi | x_1 + M_1 \notin \langle\langle \sigma_P | x_1 + M_1)_{P \neq \Psi}\rangle.$$

This is a consequence of the following:

LEMMA. *Assume* $\operatorname{char}(k) \neq 2$. *Let* $r \in \mathbf{N}$. *Consider the ring* $k[T_{ij}]$ *where* $1 \leqslant i \leqslant j \leqslant r$. *Put* $T_{ij} := T_{ji}$ *if* $i > j$. *Put* $Q := \{1, \ldots, r\}$. *If* $\varnothing \neq P \subset Q$ *define* $\sigma_P := \det(T_{ij})_{i,j \in P}$. *Then* $\sigma_Q \notin \langle\langle \sigma_P)_{P \neq Q}\rangle$.

PROOF. We may assume $r \geqslant 3$. Let I be the ideal generated by all T_{ij} such that $1 \neq |i - j| \neq r - 1$, and all T_{ij}^2. It is easy to see that $\sigma_P \notin I$ if and only if $P = Q$.

(4.3) The following facts are not proved here, see [13, pp. 11–13].

(i) The mapping $\Sigma_{(1)}$: $P_{(1)}(n) \longrightarrow P$ is injective.

(ii) If $\Sigma_1 \lambda \leqslant \Sigma_1 \mu$ where $\lambda, \mu \in P_1(n)$ then $\Sigma\lambda \leqslant \Sigma\mu$.

(iii) If $\lambda \in P_{(1)}(n)$ then $\gamma_{(1)}(\lambda) = 2|\Sigma_{(1)}\lambda|$.

Using (1.7), (2.1)(a), (3.5), (3.8), (3.10), (4.2) we get the following

COROLLARY. *Case* I *and* II (1, 0). *Let* $x \in \mathfrak{D}(\lambda)$ *and* $y \in N(G)$.

(a) $y \in \mathfrak{D}(\lambda)$ *if and only if* $\operatorname{ord}(y, N(G)/k) = \operatorname{ord}(x, N(G)/k)$.

(b) $y \in Gx$ *if and only if* $(N(G), y) \sim (N(G), x)$, *cf.* (1.7).

Assume that k is a field.

(c) $\operatorname{codim}(Gx, N(G)) = 2|\operatorname{ord}(x, N(G)/k)|$.

(d) $y \in \overline{Gx}$ *if and only if* $\operatorname{ord}(y, N(G)/k) \geqslant \operatorname{ord}(x, N(G)/k)$.

(4.4) REMARK. In (4.2) Case II $(0, \zeta)$, inequality occurs if λ_1 is even and also if $\lambda_* = (3, 3, 2, 2)$, but we have equality if $\lambda_* = (3, 3, 2, 2, 1)$. In the last case we have

$$\operatorname{codim}(Gx, N(G)) = \gamma_0(\lambda) > 2|\Sigma_0\lambda| = 2|\operatorname{ord}(x, N(G)/k)|$$

if k is a field, compare (4.3)(c) and (4.9) table B_5.

(4.5) The polynomials f_a are defined by $f_a := 0$ if $a < 0, f_0 := 1$ and $f_a := \Sigma_{i \geq 1} X_i f_{a-i}$ if $a > 0$. They are determined by the generating function

$$\sum_{a=0}^{\infty} T^a f_a = \Big(1 - \sum_{i \geq 1} X_i T^i\Big)^{-1}.$$

Clearly $f_a(X_1) = X_1^a$ if $a \geq 0$. One can prove that

$$f_a(X_1, X_2) = \sum \binom{a-i}{i} X_1^{a-2i} X_2^i \quad (0 \leq i \leq \tfrac{1}{2}a).$$

Let A^m denote the affine space over k of rank m, say with coordinate ring $k[X_1, \ldots, X_m]$. It is pointed in some point of the origin section. The *Kleinian singularities* A_l and D_l are the pointed subschemes of A^3 given by one equation:

A_l, $l \geq 1$, by $X_1^{l+1} + X_2X_3 = 0$,

D_l, $l \geq 3$, by $X_1^{l-1} - X_1X_2^2 + X_3^2 = 0$, if $1/2 \in k$.

We define the following *singularities.*

If $l \geq 3$, AA_l in $\mathrm{A}^2 \times \mathrm{A}^4$ by

$$\begin{cases} f_l(X_1, X_2) + Y_1Y_3 + Y_2Y_4 = 0, \\ X_2 f_{l-1}(X_1, X_2) - Y_4(X_1Y_2 - X_2Y_1) + Y_2Y_3 = 0. \end{cases}$$

If $1/2 \in k$ and $l \geq 3$, BB_l in $\mathrm{A}^2 \times \mathrm{A}^4$ by

$$\begin{cases} f_{l-1}(2X_1, -X_2^2) - 2Y_1Y_3 + Y_2^2 - Y_4^2 = 0, \\ X_2^2 f_{l-2}(2X_1, -X_2^2) + (Y_3 - X_1Y_1)^2 - X_2^2Y_1^2 - 2Y_4(X_1Y_4 - X_2Y_2) = 0. \end{cases}$$

If $1/2 \in k$ and $l \geq 2$, CC_l in $\mathrm{A}^3 \times \mathrm{A}^2$ by

$$(X_3^2 - X_1X_2)^l + X_1Y_1^2 + 2X_3Y_1Y_2 + X_2Y_2^2 = 0.$$

If $1/2 \in k$ and $l \geq 5$, CD_l in $\mathrm{A}^2 \times \mathrm{A}^4$ by

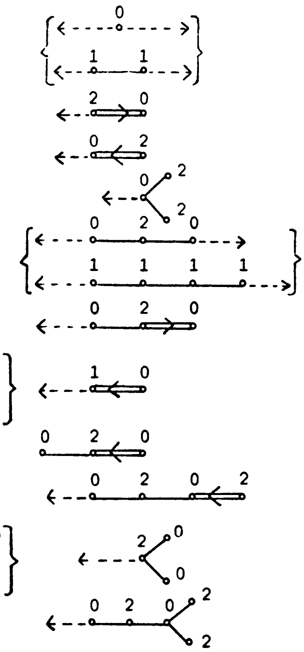
$$\begin{cases} f_{l-2}(X_1, X_2) + X_1Y_2^2 - X_2Y_1^2 - Y_3^2 + 2Y_2Y_4 = 0, \\ X_2 f_{l-3}(X_1, X_2) + X_2(X_1Y_1^2 + Y_2^2 - 2Y_1Y_3) + Y_4^2 = 0. \end{cases}$$

If $1/2 \in k$ and $l \geq 3$, DD_l in $\mathrm{A}^3 \times \mathrm{A}^3$ by

$$\begin{cases} (x_1^2 + x_2^2 + x_3^2)^{l-1} + y_1^2 + y_2^2 + y_3^2 = 0, \\ x_1y_1 + x_2y_2 + x_3y_3 = 0. \end{cases}$$

(4.6) PROPOSITION. *Assume in Case* II *that* $l + \epsilon + \zeta \geqslant 3$. *Consider* $\lambda \in P_{(\epsilon)}(n)$ *with* $0 < \gamma_{(\epsilon)}(\lambda) < 6$, *cf.* (3.8). *If* $x \in \mathfrak{O}(\lambda)$ *then* $(N(G), x)$ *is smoothly equivalent* (*cf.* (1.7)) *to the singularity* (*cf.* (4.5)) *given in the following table.*

G		λ_*	Dynkin diagram	$\gamma_{(\epsilon)}(\lambda)$	singularity
Gl_n	$n \geqslant 2$	$(n-1,1)$		2	A_{n-1}
$SO_{2\ell+1}$	$\ell \geqslant 2$	$(2\ell-1,1,1)$		2	$A_{2\ell-1}$
$Sp_{2\ell}$	$\ell \geqslant 2$	$(2\ell-2,2)$		2	$D_{\ell+1}$
$SO_{2\ell}$	$\ell \geqslant 3$	$(2\ell-3,3)$		2	D_ℓ
Gl_n	$n \geqslant 4$	$(n-2,2)$		4	AA_{n-1}
$SO_{2\ell+1}$	$\ell \geqslant 3$	$(2\ell-3,3,1)$		4	BB_ℓ
	$\ell = 2$	$(2,2,1)$		4	CC_ℓ
$Sp_{2\ell}$	$\ell \geqslant 2$	$(2\ell-2,1,1)$			
	$\ell = 3$	$(3,3)$		4	DD_4
	$\ell \geqslant 4$	$(2\ell-4,4)$		4	$CD_{\ell+1}$
$SO_{2\ell}$	$\ell \geqslant 3$	$(2\ell-3,1,1,1)$		4	DD_ℓ
	$\ell = 4$	$(4,4)$			
	$\ell \geqslant 5$	$(2\ell-5,5)$		4	CD_ℓ



REMARK. We have $\gamma_{(\epsilon)}(\lambda) = \text{codim}(Gx, N(G)_{(k(x))})$. For the singularities with $\gamma_{(\epsilon)}(\lambda) = 2$, compare [5] and [21, pp. 140–158]. In the table we have added the Dynkin diagram of the section $x \in \mathfrak{g}(k(x))$, cf. [20, III, IV], where ←––– means a string with numbers 2 attached to the nodes.

PROOF. The classification of all possibilities for λ is easy. By the sequence of reductions used in (4.2)(1) we may assume that $x = z(\mathfrak{p})$ where z is a standard nilpotent with partition λ and $\mathfrak{p} \in \text{Spec}(k)$. In Case II the base-data for z may be prescribed within the bounds set by (3.3)(2). Let $z + L$ be the cross section of (3.7). Then $(z + L) \cap N(G)$ is a cross section at z for the action of G on $N(G)$. So $(N(G), x)$ is smoothly equivalent to $((z + L) \cap N(G), z(\mathfrak{p}))$ by (2.1) (a). The two singularities to be determined for Gl_n will be examples in (4.7) and (4.8). We do not give the tedious calculations needed to settle Case II, see [13, p. 79] for some indications.

(4.7) *Case* I *with* $\lambda_* = (p, 1^q)$, i.e. $(p, 1, \ldots, 1)$ with q times 1. We have $n = p + q$ and $r := \lambda^1 = q + 1$. On $z + L$ we define the coordinate functions ξ_a, ξ_{ij} as follows: if R is a k-algebra and $x \in (z + L)(R)$, then

(1) $$x = z - \sum_{a=1}^{p} \xi_a(x)e(1, p-a; 1, p-1) - \sum_{(i,j)\neq(1,1)} \xi_{ij}(x)e(i, 0; j, \lambda_j - 1).$$

So $A(z+L) = k[\xi_a, \xi_{ji}]$. Put $\xi_{11} = 0$.

If $a \geqslant 1$, let $s_a, h_a \in k[\xi_{ij}]$ be defined by

(2) $$\begin{cases} s_a = \sum \det(\xi_{ij})_{i,j\in I} \\ h_a = \sum \det(\xi_{ij})_{i,j\in\{1\}\cup I} \end{cases}$$

where in both cases the summation is over the subsets I of $\{2, \ldots, r\}$ with $\#I = a$.

Clearly, if $a \geqslant r$ then $s_a = h_a = 0$. The subscheme $(z+L) \cap N(G)$ of $z+L$ is defined by the equations $\sigma_m|(z+L) = 0$ $(1 \leqslant m \leqslant n)$. One verifies that

(3) $$\begin{cases} (-1)^m \sigma_m|(z+L) = \xi_m + s_m + \sum_{a=1}^{m-1} \xi_a s_{m-a} & \text{if } 1 \leqslant m \leqslant p, \\ (-1)^m \sigma_m|(z+L) = h_{m-p} + s_m + \sum_{a=1}^{p} \xi_a s_{m-a} & \text{if } p < m \leqslant n. \end{cases}$$

The first p equations can be solved inductively. With the notations of (4.5) we obtain $\xi_m = f_m(-s_1, -s_2, \ldots, -s_q)$ $(1 \leqslant m \leqslant p)$. So $(z+L) \cap N(G)$ is isomorphic to the subscheme of Spec $k[\xi_{ij}]$ defined by the equations

(4) $$h_{m-p} + \sum_{a=0}^{p} s_{m-a} f_a(-s_1, \ldots, -s_q) = 0 \qquad (p < m \leqslant n).$$

EXAMPLES. (a) $\lambda_* = (n-1, 1)$. Putting $X_1 = -\xi_{22}, X_2 = \xi_{12}, X_3 = \xi_{21}$, we get the singularity A_{n-1}, cf. (4.5).

(b) $\lambda_* = (n-2, 1, 1)$. The scheme $(z+L) \cap N(G)$ is isomorphic to the subscheme of $\mathbf{A}^8 = \mathrm{Spec}(k[\xi_{ij}])$, where $1 \leqslant i, j \leqslant 3 \leqslant i+j$, defined by the equations

(5) $$\begin{cases} f_{n-1}(-s_1, -s_2) - h_1 = 0, \\ s_2 f_{n-2}(-s_1, -s_2) + h_2 = 0, \end{cases}$$

where

$$s_1 = \xi_{22} + \xi_{33}, \quad s_2 = \xi_{22}\xi_{33} - \xi_{23}\xi_{32}, \quad h_1 = \xi_{12}\xi_{21} + \xi_{13}\xi_{31}, \quad \text{and} \quad h_2 = \begin{vmatrix} 0 & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{vmatrix}.$$

(4.8) *Case* I *for arbitrary* λ. We use a different cross section, viz. $z + L''$ defined by $L'' := \Sigma_{i,j} L''_{ij}$ where $L''_{ij} := L_{ij}$ if $i \neq 1$ or $j = 1$ and $L''_{1,j}(R) := \Sigma_{0<b<\lambda_j} Re(1, 0; j, b)$ if $j \neq 1$, see (3.7). Again we have

$$(N(G), x) \sim ((z + L'') \cap N(G), z(\mathfrak{p})).$$

Put $p := \lambda_1$, $q := n - \lambda_1$ and $\mu_* := (p, 1^q)$. Put $z' := \Sigma_{a=0}^{p-2} e(1, a+1; 1, a)$, so that z' is a standard nilpotent element in $\mathfrak{g}(k)$ with partition μ. In the obvious way we define base-data $(\mathfrak{f}', \mu)$ for z'. The cross section $z' + L'$ at z' used in (4.7) contains $z + L''$. So we can use the elimination in (4.7) of ξ_a, $1 \leqslant a \leqslant p$, substituting into the matrix (ξ_{ij}) at some places the constant functions 0 or -1, cf. (4.7)(1).

EXAMPLE. If $\lambda_* = (n-2, 2)$, $n \geqslant 4$, we use the matrix

$$(\xi_{ij}) = \begin{pmatrix} 0 & Y_3 & Y_4 \\ Y_1 & -X_1 & -1 \\ Y_2 & -X_2 & 0 \end{pmatrix}$$

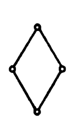
and we obtain the equations (4.7)(5) where $s_1 = -X_1$, $s_2 = -X_2$, $h_1 = Y_1 Y_3 + Y_2 Y_4$ and $h_2 = Y_4(X_1 Y_2 - X_2 Y_1) - Y_2 Y_3$. So $(z + L'') \cap N(G)$ is isomorphic to the singularity AA_{n-1}, cf. (4.5).

(4.9) *Tables for the orbits in $N(G)$.* We give the adjacency structure (cf. (3.10)), the Dynkin diagram (cf. [20, IV]), the codimension of the orbits $\gamma_{(\epsilon)}(\lambda)$ (cf. (3.8)), and the partition ord $=$ ord$(x, N(G)/k)$ (cf. (4.2)). The number of orbits is denoted by #. In the cases SO_{2l} with even l, the partition λ may represent two orbits, cf. (3.5). We give the Dynkin diagram of one of them and indicate how to get the other one by the symbol ↷ .

For SO_n we give $\Sigma_0\lambda$, which is a lower bound of ord, cf. (4.2). Whenever there are reasons to assume ord $\neq \Sigma_0\lambda$, we give a conjectured value of ord or a question mark. As $D_2 = A_1 + A_1$, $B_2 = C_2$ and $D_3 = A_3$, the values of ord for the cases SO_4, SO_5 and SO_6 are not conjectural.

A_1	Gl_2	λ_*	Dy ∘	$\gamma(\lambda)$	ord_*
		2	2	0	0
# = 2		1 1	0	2	1

A_2	Gl_3	λ_*	Dy ∘—∘	$\gamma(\lambda)$	ord_*
		3	2 2	0	0
		2 1	1 1	2	1
# = 3		1 1 1	0 0	6	2 1

$D_2 = A_1+A_1$	SO_4	λ_*	Dy ∘ ∘	$\gamma_0(\lambda)$	$(\Sigma_0\lambda)_*$	ord_*
		3 1	2 2	0	0	
		2 2	0 ↷ 2	2	0	1
# = 4		1^4	0 0	4	1 1	

$B_2 = C_2$	SO_5	λ_*	Dy	$\gamma_0(\lambda)$	$(\Sigma_0\lambda)_*$	ord_*
		5	2 2	0	0	0
		3 1 1	2 0	2	1	1
		2 2 1	0 1	4	1	2
# = 4		1^5	0 0	8	3 1	3 1

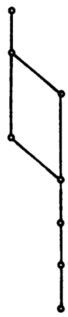
$C_2 = B_2$	Sp_4	λ_*	Dy	$\gamma_1(\lambda)$	ord_*
		4	2 2	0	0
		2 2	0 2	2	1
		2 1 1	1 0	4	2
# = 4		1^4	0 0	8	3 1

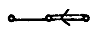
$A_3 = D_3$	Gl_4	λ_*		$\gamma(\lambda)$	ord_*
		4	2 2 2	0	0
		3 1	2 0 2	2	1
		2 2	0 2 0	4	1 1
		2 1 1	1 0 1	6	2 1
# = 5		1^4	0 0 0	12	3 2 1

$D_3 = A_3$	SO_6	λ_*		$\gamma_0(\lambda)$	$(\Sigma_0\lambda)_*$	ord_*
		5 1	2 2 2	0	0	
		3 3	0 2 2	2	1	
		3 1 1 1	2 0 0	4	1 1	
		2 2 1 1	0 1 1	6	1 1	2 1
# = 5		1^6	0 0 0	12	3 2 1	

B_3	SO_7	λ_*		$\gamma_0(\lambda)$	$(\Sigma_0\lambda)_*$	ord_*?
		7	2 2 2	0	0	
		5 1 1	2 2 0	2	1	
		3 3 1	0 2 0	4	1 1	
		3 2 2	1 0 1	6	2 1	
		3 1^4	2 0 0	8	3 1	
		2 2 1^3	0 1 0	10	3 1	3 2
# = 7		1^7	0 0 0	18	5 3 1	

C_3 Sp_6



λ_*				$\gamma_1(\lambda)$	ord_*
6	2	2	2	0	0
4 2	2	0	2	2	1
4 1 1	2	1	0	4	2
3 3	0	2	0	4	1 1
2 2 2	0	0	2	6	2 1
2 2 1 1	0	1	0	8	3 1
$2\ 1^4$	1	0	0	12	4 2
1^6	0	0	0	18	5 3 1

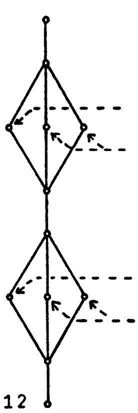
= 8

A_4 Gl_5

λ_*					$\gamma(\lambda)$	ord_*
5	2	2	2	2	0	0
4 1	2	1	1	2	2	1
3 2	1	1	1	1	4	1 1
3 1 1	2	0	0	2	6	2 1
2 2 1	0	1	1	0	8	2 1 1
$2\ 1^3$	1	0	0	1	12	3 2 1
1^5	0	0	0	0	20	4 3 2 1

= 7

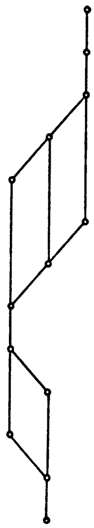
D_4 SO_8

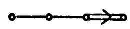


λ_*					$\gamma_0(\lambda)$	$(\Sigma_0\lambda)_*$	ord_*?
7 1	2	2	2	2	0	0	
5 3	2	0	2	2	2	1	
5 1 1 1	2	2	0	0	4	1 1	
4 4	0	2	0 ↔	2	4	1	1 1
3 3 1 1	0	2	0	0	6	1 1 1	
3 2 2 1	1	0	1	1	8	2 1 1	
$3\ 1^5$	2	0	0	0	12	3 2 1	
2^4	0	0	0 ↔	2	12	2 1 1	3 2 1
$2\ 2\ 1^4$	0	1	0	0	14	3 2 1	3 2 2
1^8	0	0	0	0	24	5 3 3 1	

= 12

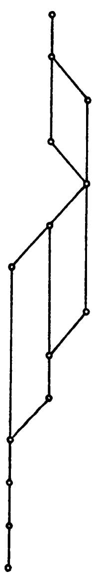
B_4 SO_9

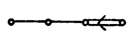


λ_*		$\gamma_0(\lambda)$	$(\Sigma_0\lambda)_*$	ord_*?
9	2 2 2 2	0	0	
7 1 1	2 2 2 0	2	1	
5 3 1	2 0 2 0	4	1 1	
5 2 2	2 1 0 1	6	2 1	
$5\ 1^4$	2 2 0 0	8	3 1	
4 4 1	0 2 0 1	6	1 1	2 1
3 3 3	0 0 2 0	8	2 1 1	
$3\ 3\ 1^3$	0 2 0 0	10	3 1 1	
3 2 2 1 1	1 0 1 0	12	3 2 1	
$3\ 1^6$	2 0 0 0	18	5 3 1	
$2^4 1$	0 0 0 1	16	3 2 1	3 3 2
$2\ 2\ 1^5$	0 1 0 0	20	5 3 1	5 3 2
1^9	0 0 0 0	32	7 5 3 1	

= 13

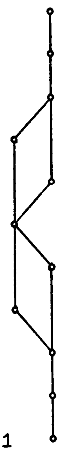
C_4 Sp_8



λ_*		$\gamma_1(\lambda)$	ord_*
8	2 2 2 2	0	0
6 2	2 2 0 2	2	1
6 1 1	2 2 1 0	4	2
4 4	0 2 0 2	4	1 1
4 2 2	2 0 0 2	6	2 1
4 2 1 1	2 0 1 0	8	3 1
$4\ 1^4$	2 1 0 0	12	4 2
3 3 2	0 1 1 0	8	2 1 1
3 3 1 1	0 2 0 0	10	3 1 1
2 2 2 2	0 0 0 2	12	3 2 1
2 2 2 1 1	0 0 1 0	14	4 2 1
$2\ 2\ 1^4$	0 1 0 0	18	5 3 1
$2\ 1^6$	1 0 0 0	24	6 4 2
1^8	0 0 0 0	32	7 5 3 1

= 14

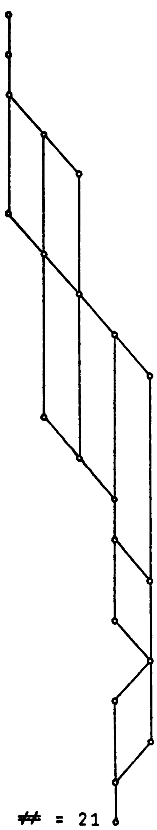
A_5 Gl_6



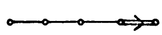
$\# = 11$

λ_*						$\gamma(\lambda)$	ord_*
6	2	2	2	2	2	0	0
5 1	2	2	0	2	2	2	1
4 2	2	0	2	0	2	4	1 1
4 1 1	2	1	0	1	2	6	2 1
3 3	0	2	0	2	0	6	1 1 1
3 2 1	1	1	0	1	1	8	2 1 1
3 1 1 1	2	0	0	0	2	12	3 2 1
2 2 2	0	0	2	0	0	12	2 2 1 1
2 2 1 1	0	1	0	1	0	14	3 2 1 1
2 1^4	1	0	0	0	1	20	4 3 2 1
1^6	0	0	0	0	0	30	5 4 3 2 1

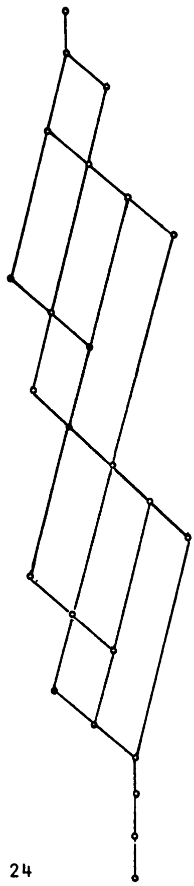
B_5 SO_{11}



$\# = 21$

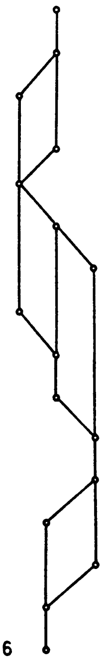
λ_*						$\gamma_0(\lambda)$	$(\Sigma_0\lambda)_*$	ord_*?
11	2	2	2	2	2	0	0	
9 1 1	2	2	2	2	0	2	1	
7 3 1	2	2	0	2	0	4	1 1	
7 2 2	2	2	1	0	1	6	2 1	
7 1 1 1 1	2	2	2	0	0	8	3 1	
5 5 1	0	2	0	2	0	6	1 1 1	
5 3 3	2	0	0	2	0	8	2 1 1	
5 3 1 1 1	2	0	2	0	0	10	3 1 1	
5 2 2 1 1	2	1	0	1	0	12	3 2 1	
5 1^6	2	2	0	0	0	18	5 3 1	
4 4 3	0	1	1	0	1	10	2 1 1	2 2 1
4 4 1 1 1	0	2	0	1	0	12	3 1 1	3 2 1
3 3 3 1 1	0	0	2	0	0	14	3 2 1 1	
3 3 2 2 1	0	1	0	1	0	16	3 2 1 1	= ord_*, cf. (4.4)
3 3 1^5	0	2	0	0	0	20	5 3 1 1	
3 2 2 2 2	1	0	0	0	1	20	4 3 2 1	
3 2 2 1^4	1	0	1	0	0	22	5 3 2 1	
3 1^8	2	0	0	0	0	32	7 5 3 1	
2 2 2 2 1^3	0	0	0	1	0	26	5 3 2 1	5 3 3 2
2 2 1^7	0	1	0	0	0	34	7 5 3 1	7 5 3 2
1^{11}	0	0	0	0	0	50	9 7 5 3 1	

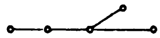
C_5 Sp_{10}



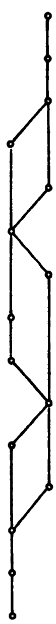
λ_*						$\gamma_1(\lambda)$	ord$_*$
10	2	2	2	2	2	0	0
8 2	2	2	2	0	2	2	1
8 1 1	2	2	2	1	0	4	2
6 4	2	0	2	0	2	4	1 1
6 2 2	2	2	0	0	2	6	2 1
6 2 1 1	2	2	0	1	0	8	3 1
6 1 1 1 1	2	2	1	0	0	12	4 2
5 5	0	2	0	2	0	6	1 1 1
4 4 2	0	2	0	0	2	8	2 1 1
4 4 1 1	0	2	0	1	0	10	3 1 1
4 3 3	1	0	1	1	0	10	2 2 1
4 2 2 2	2	0	0	0	2	12	3 2 1
4 2 2 1 1	2	0	0	1	0	14	4 2 1
$4\ 2\ 1^4$	2	0	1	0	0	18	5 3 1
$4\ 1^6$	2	1	0	0	0	24	6 4 2
3 3 2 2	0	1	0	1	0	14	3 2 1 1
3 3 2 1 1	0	1	1	0	0	16	4 2 1 1
$3\ 3\ 1^4$	0	2	0	0	0	20	5 3 1 1
2 2 2 2 2	0	0	0	0	2	20	4 3 2 1
$2^4 1\ 1$	0	0	0	1	0	22	5 3 2 1
$2^3 1^4$	0	0	1	0	0	26	6 4 2 1
$2\ 2\ 1^6$	0	1	0	0	0	32	7 5 3 1
$2\ 1^8$	1	0	0	0	0	40	8 6 4 2
1^{10}	0	0	0	0	0	50	9 7 5 3 1

= 24



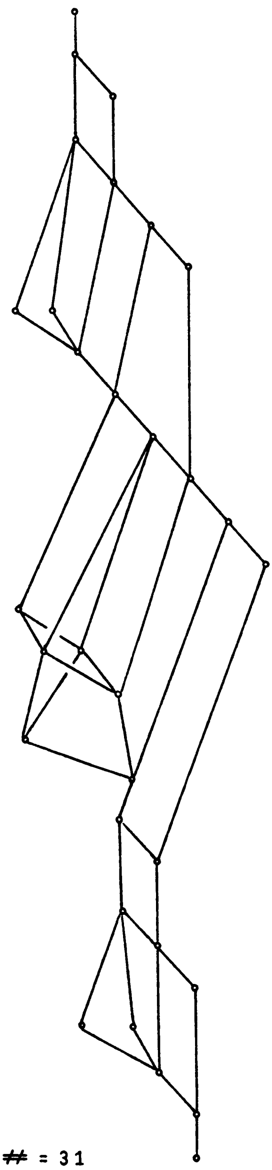


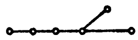
D_5	SO_{10}	λ_*						$\gamma_0(\lambda)$	$(\Sigma_0\lambda)_*$	ord_*?
		9 1	2	2	2	2	2	0	0	
		7 3	2	2	0	2	2	2	1	
		7 1 1 1	2	2	2	0	0	4	1 1	
		5 5	0	2	0	2	2	4	1 1	
		5 3 1 1	2	0	2	0	0	6	1 1 1	
		5 2 2 1	2	1	0	1	1	8	2 1 1	
		$5\ 1^5$	2	2	0	0	0	12	3 2 1	
		4 4 1 1	0	2	0	1	1	8	1 1 1	2 1 1
		3 3 3 1	0	0	2	0	0	10	2 1 1 1	
		3 3 2 2	0	1	0	1	1	12	2 1 1 1	3 1 1 1
		$3\ 3\ 1^4$	0	2	0	0	0	14	3 2 1 1	
		$3\ 2\ 2\ 1^3$	1	0	1	0	0	16	3 2 2 1	
		$3\ 1^7$	2	0	0	0	0	24	5 3 3 1	
		$2^4 1\ 1$	0	0	0	1	1	20	3 2 2 1	3 3 2 2
		$2\ 2\ 1^6$	0	1	0	0	0	26	5 3 3 1	5 3 3 2
# = 16		1^{10}	0	0	0	0	0	40	7 5 4 3 1	



A_6	Gl_7	λ_*							$\gamma(\lambda)$	ord_*
		7	2	2	2	2	2	2	0	0
		6 1	2	2	1	1	2	2	2	1
		5 2	2	1	1	1	1	2	4	1 1
		5 1 1	2	2	0	0	2	2	6	2 1
		4 3	1	1	1	1	1	1	6	1 1 1
		4 2 1	2	0	1	1	0	2	8	2 1 1
		$4\ 1^3$	2	1	0	0	1	2	12	3 2 1
		3 3 1	0	2	0	0	2	0	10	2 1 1 1
		3 2 2	1	0	1	1	0	1	12	2 2 1 1
		3 2 1 1	1	1	0	0	1	1	14	3 2 1 1
		$3\ 1^4$	2	0	0	0	0	2	20	4 3 2 1
		2 2 2 1	0	0	1	1	0	0	18	3 2 2 1 1
		$2\ 2\ 1^3$	0	1	0	0	1	0	22	4 3 2 1 1
		$2\ 1^5$	1	0	0	0	0	1	30	5 4 3 2 1
# = 15		1^7	0	0	0	0	0	0	42	6 5 4 3 2 1

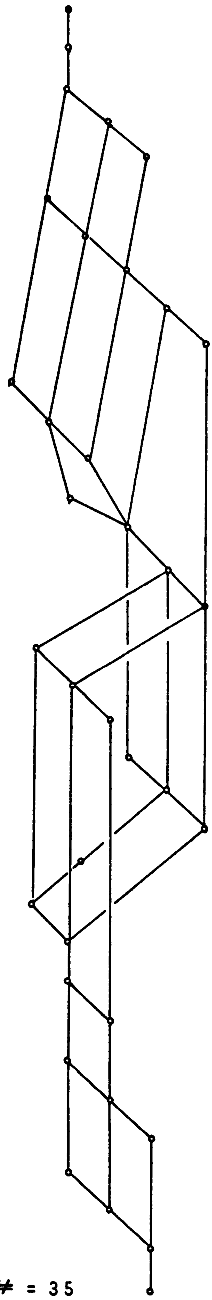
D_6 SO_{12}



λ_*		$\gamma_0(\lambda)$	$(\Sigma_0\lambda)_*$	ord$_*$?
11,1	2 2 2 2 2 2	0	0	
9 3	2 2 2 0 2 2	2	1	
9 1 1 1	2 2 2 2 0 0	4	1 1	
7 5	2 0 2 0 2 2	4	1 1	
7 3 1 1	2 2 0 2 0 0	6	1 1 1	
7 2 2 1	2 2 1 0 1 1	8	2 1 1	
7 1^5	2 2 2 0 0 0	12	3 2 1	
6 6	0 2 0 2 0 2	6	1 1	1 1 1
5 5 1 1	0 2 0 2 0 0	8	1 1 1 1	
5 3 3 1	2 0 0 2 0 0	10	2 1 1 1	
5 3 2 2	2 0 1 0 1 1	12	2 1 1 1	?
5 3 1^4	2 0 2 0 0 0	14	3 2 1 1	
5 2 2 1 1 1	2 1 0 1 0 0	16	3 2 2 1	
5 1^7	2 2 0 0 0 0	24	5 3 3 1	
4 4 3 1	0 1 1 0 1 1	12	2 1 1 1	?
4 4 2 2	0 2 0 0 0 2	14	2 1 1 1	?
4 4 1^4	0 2 0 1 0 0	16	3 2 1 1	?
3 3 3 3	0 0 0 2 0 0	16	3 2 1 1 1	
3 3 3 1^3	0 0 2 0 0 0	18	3 2 2 1 1	
3 3 2 2 1 1	0 1 0 1 0 0	20	3 2 2 1 1	?
3 3 1^6	0 2 0 0 0 0	26	5 3 3 1 1	
3 $2^4$1	1 0 0 0 1 1	24	4 3 2 2 1	
3 2 2 1^5	1 0 1 0 0 0	28	5 3 3 2 1	
3 1^9	2 0 0 0 0 0	40	7 5 4 3 1	
2 2 2 2 2 2	0 0 0 0 0 2	30	4 3 2 2 1	?
2 2 2 2 1^4	0 0 0 1 0 0	32	5 3 3 2 1	?
2 2 1^8	0 1 0 0 0 0	42	7 5 4 3 1	?
1^{12}	0 0 0 0 0 0	60	9 7 5 5 3 1	

= 31

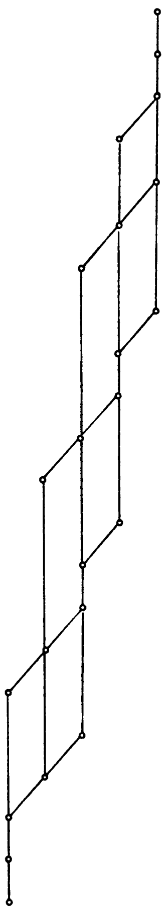
B_6 SO_{13}



$\# = 35$

λ_*		$\gamma_0(\lambda)$	$(\Sigma_0\lambda)_*$	$\mathrm{ord}_*?$
13	2 2 2 2 2 2	0	0	
11,1,1	2 2 2 2 2 0	2	1	
9 3 1	2 2 2 0 2 0	4	1 1	
9 2 2	2 2 2 1 0 1	6	2 1	
9 1 1 1 1	2 2 2 2 0 0	8	3 1	
7 5 1	2 0 2 0 2 0	6	1 1 1	
7 3 3	2 2 0 0 2 0	8	2 1 1	
7 3 1 1 1	2 2 0 2 0 0	10	3 1 1	
7 2 2 1 1	2 2 1 0 1 0	12	3 2 1	
$7\ 1^6$	2 2 2 0 0 0	18	5 3 1	
6 6 1	0 2 0 2 0 1	8	1 1 1	2 1 1
5 5 3	0 2 0 0 2 0	10	2 1 1 1	
5 5 1 1 1	0 2 0 2 0 0	12	3 1 1 1	
5 4 4	1 0 1 1 0 1	12	2 2 1 1	
5 3 3 1 1	2 0 0 2 0 0	14	3 2 1 1	
5 3 2 2 1	2 0 1 0 1 0	16	3 2 1 1	?
$5\ 3\ 1^5$	2 0 2 0 0 0	20	5 3 1 1	
5 2 2 2 2	2 1 0 0 0 1	20	4 3 2 1	
$5\ 2\ 2\ 1^4$	2 1 0 1 0 0	22	5 3 2 1	
$5\ 1^8$	2 2 0 0 0 0	32	7 5 3 1	
4 4 3 1 1	0 1 1 0 1 0	16	3 2 1 1	3 2 2 1
4 4 2 2 1	0 2 0 0 0 1	18	3 2 1 1	3 3 2 1
$4\ 4\ 1^5$	0 2 0 1 0 0	22	5 3 1 1	5 3 2 1
3 3 3 3 1	0 0 0 2 0 0	20	3 3 2 1 1	
3 3 3 2 2	0 0 1 0 1 0	22	4 3 2 1 1	
$3\ 3\ 3\ 1^4$	0 0 2 0 0 0	24	5 3 2 1 1	
$3\ 3\ 2\ 2\ 1^3$	0 1 0 1 0 0	26	5 3 2 1 1	?
$3\ 3\ 1^7$	0 2 0 0 0 0	34	7 5 3 1 1	
$3\ 2^4 1\ 1$	1 0 0 0 1 0	30	5 4 3 2 1	
$3\ 2\ 2\ 1^6$	1 0 1 0 0 0	36	7 5 3 2 1	
$3\ 1^{10}$	2 0 0 0 0 0	50	9 7 5 3 1	
$2^6 1$	0 0 0 0 0 1	36	5 4 3 2 1	?
$2^4 1^5$	0 0 0 1 0 0	40	7 5 3 2 1	?
$2\ 2\ 1^9$	0 1 0 0 0 0	52	9 7 5 3 1	?
1^{13}	0 0 0 0 0 0	72	11,9,7,5,3,1	

A_7 Gl_8



$\# = 22$

λ_*		$\gamma(\lambda)$	ord_*
8	2 2 2 2 2 2 2	0	0
7 1	2 2 2 0 2 2 2	2	1
6 2	2 2 0 2 0 2 2	4	1 1
6 1 1	2 2 1 0 1 2 2	6	2 1
5 3	2 0 2 0 2 0 2	6	1 1 1
5 2 1	2 1 1 0 1 1 2	8	2 1 1
5 1 1 1	2 2 0 0 0 2 2	12	3 2 1
4 4	0 2 0 2 0 2 0	8	1 1 1 1
4 3 1	1 1 1 0 1 1 1	10	2 1 1 1
4 2 2	2 0 0 2 0 0 2	12	2 2 1 1
4 2 1 1	2 0 1 0 1 0 2	14	3 2 1 1
$4\ 1^4$	2 1 0 0 0 1 2	20	4 3 2 1
3 3 2	0 1 1 0 1 1 0	14	2 2 1 1 1
3 3 1 1	0 2 0 0 0 2 0	16	3 2 1 1 1
3 2 2 1	1 0 1 0 1 0 1	18	3 2 2 1 1
$3\ 2\ 1^3$	1 1 0 0 0 1 1	22	4 3 2 1 1
$3\ 1^5$	2 0 0 0 0 0 2	30	5 4 3 2 1
2 2 2 2	0 0 0 2 0 0 0	24	3 3 2 2 1 1
2 2 2 1 1	0 0 1 0 1 0 0	26	4 3 2 2 1 1
$2\ 2\ 1^4$	0 1 0 0 0 1 0	32	5 4 3 2 1 1
$2\ 1^6$	1 0 0 0 0 0 1	42	6 5 4 3 2 1
1^8	0 0 0 0 0 0 0	56	7 6 5 4 3 2 1

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MATHEMATISCH INSTITUUT DER RIJKSUNIVERSITEIT TE UTRECHT, UTRECHT, THE NETHERLANDS

Current address: Mathematisch Instituut, Postbus 800, Groningen, The Netherlands